PERGAMON

International Journal of Solids and Structures 40 (2003) 6347–6367

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

Equilibrium conditions of a tensegrity structure

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Received 13 March 2003; received in revised form 7 July 2003

Abstract

This paper characterizes the necessary and sufficient conditions for tensegrity equilibria. Static models of tensegrity structures are reduced to linear algebra problems, after first characterizing the problem in a vector space where direction cosines are not needed. This is possible by describing the components of all member vectors. While our approach enlarges (by a factor of 3) the vector space required to describe the problem, the advantage of enlarging the vector space makes the mathematical structure of the problem amenable to linear algebra treatment. Using the linear algebraic techniques, many variables are eliminated from the final existence equations.

Keywords: Tensegrity; Structure; Statics; Equilibrium conditions; Linear algebra

1. Introduction

The tensegrity structures introduced by Snelson (1996) pose a wonderful blend of geometry and mechanics. In addition, they have engineering appeal in problems requiring large changes is structural shape. Tensegrity structures exist as a prestressed stable connection of bars and strings. Most existing *smart structure* methods are limited to small displacements, but the control of tensegrity structures allows very large shape changes to occur (Skelton and Sultan, 1997; Skelton et al., 2001a; Motro, 1992). Therefore, an efficient set of analytical tools could be the enabler to a host of new engineering concepts for deployable and shape controllable structures. This paper characterizes the static equilibria of tensegrity structures in terms of vectors which describe the elements (bars and strings), thereby eliminating the need to use direction cosines and the subsequent transcendental functions that follow their use. For a comparison of previous methods of form-finding in tensegrity structures, see Tibert and Pellegrino (2001), Vassart and Motro (1999), Motro et al. (1994), Linkwitz (1999), Barnes (1998), and Schek (1974).

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0020-7683/$ - see front matter
doi:10.1016/S0020-7683(03)00400-1

It is well known in a variety of mathematical problems that enlarging the domain in which the problem is posed can often simplify the mathematical treatment. In fact, many nonlinear problems admit solutions by linear techniques by enlarging the domain of the problem. For example, nonlinear Riccati equations are known to be solvable by linear algebra in a space that is twice the size of the original problem statement. The purpose of this paper is to show that the mathematical structure of the equations admits some treatments by linear algebra methods by enlarging the vector space in which the tensegrity statics problem is characterized.

Our results characterize the equilibria conditions of tensegrity structures in terms of a very small number of variables since the necessary and sufficient conditions of the linear algebra treatment allow the elimination of several of the original variables. These results can be used for efficient algorithms to design and simulate a large class of tensegrity structures. Tensegrity concepts have been around for 50 years without efficient design procedures (Kenner, 1976; Pugh, 1976; Connelly, 1982, 1993, 1999; Ingber, 1993, 1997, 1998; Williamson and Skelton, 1998a,b; Motro, 1984, 1990, 2001; Skelton et al., 2001b).

The paper is organized as follows. After the review of mathematical preliminaries in Section 2, Section 3 introduces the network representations of tensegrity structures as an oriented graph in real three dimensional space. Geometric connectivity, equilibrium, and a coordinate transformation will be introduced. Section 4 introduces the algebraic equilibrium conditions. After we derive necessary and sufficient conditions for the existence of an unloaded tensegrity structure in equilibrium, we write the necessary and sufficient conditions for the externally loaded structure in equilibrium. A couple of examples will illustrate the results.

2. Algebraic preliminaries

We let $\mathbf{I}_n$ define the $n \times n$ identity matrix, and $\mathbf{0}$ define an $n \times m$ matrix of zeros. (The dimensions of $\mathbf{0}$ will be clear from the context.) We also let $\rho(\mathbf{A})$ define the rank of the matrix $\mathbf{A}$. Let $\mathbf{A} \in \Re^{m\times n}$ and $\mathbf{B} \in \Re^{p\times q}$, then the Kronecker product (Horn and Johnson, 1985) of $\mathbf{A}$ and $\mathbf{B}$ is defined as

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} A(1,1)\mathbf{B} & A(1,2)\mathbf{B} & \cdots & A(1,n)\mathbf{B} \\ A(2,1)\mathbf{B} & A(2,2)\mathbf{B} & \cdots & A(2,n)\mathbf{B} \\ \vdots & \vdots & & \vdots \\ A(m,1)\mathbf{B} & A(m,2)\mathbf{B} & \cdots & A(m,n)\mathbf{B} \end{bmatrix} \in \Re^{mp\times nq}$$

where $A(i,j)$ is the (i,j) element of a matrix $\mathbf{A}$. Then we have the following result.

Lemma 1. *The following statements are true.*

(i) *Suppose* $\mathbf{A} \in \Re^{n\times r}$, $\mathbf{B} \in \Re^{n\times r}$, $\mathbf{C} \in \Re^{r\times p}$. *Then*

$$(\mathbf{A} \otimes \mathbf{I}_m)^{\mathrm{T}} = \mathbf{A}^{\mathrm{T}} \otimes \mathbf{I}_m$$

$$\mathbf{A} \otimes \mathbf{I}_m + \mathbf{B} \otimes \mathbf{I}_m = (\mathbf{A} + \mathbf{B}) \otimes \mathbf{I}_m$$

$$(\mathbf{A} \otimes \mathbf{I}_m)(\mathbf{C} \otimes \mathbf{I}_m) = (\mathbf{AC}) \otimes \mathbf{I}_m$$

$$\mathrm{rank}(\mathbf{A} \otimes \mathbf{I}_m) = m \times \mathrm{rank}(\mathbf{A}).$$

(ii) *Suppose* $\mathbf{A} \in \Re^{n\times n}$ *has eigenvalues* $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. *Then* $\mathbf{A} \otimes \mathbf{I}_m$ *also has eigenvalues* $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ *where each eigenvalue is repeated m times.*

The derivations in this paper rely heavily on the *singular value decomposition* $\mathrm{svd}(\mathbf{A}) = \{\mathbf{U}_A, \mathbf{\Sigma}_A, \mathbf{V}_A\}$ of a matrix $\mathbf{A}$ as expressed in the following result (Horn and Johnson, 1985).

Lemma 2

(i) *Suppose an* $n \times m$ *matrix* $\mathbf{A}$ *has rank* r_A, *then there exists an* $n \times n$ *unitary matrix* $\mathbf{U}_A$, *an* $m \times m$ *unitary matrix* $\mathbf{V}_A$ *and a positive definite* $r_A \times r_A$ *diagonal matrix* $\mathbf{\Sigma}_{1A}$ *such that*

$$\mathbf{A} = \mathbf{U}_A \mathbf{\Sigma}_A \mathbf{V}_A^{\mathrm{T}}, \quad \mathbf{\Sigma}_A = \begin{bmatrix} \mathbf{\Sigma}_{1A} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}. \tag{1}$$

(ii) *If* $\{\mathbf{U}_A, \mathbf{V}_A\}$ *are partitioned such that*

$$\mathbf{U}_A = [\mathbf{U}_{1A}, \mathbf{U}_{2A}], \quad \mathbf{V} = [\mathbf{V}_{1A}, \mathbf{V}_{2A}] \tag{2}$$

with

$$\mathbf{U}_{1A} \in \Re^{n \times r_A}, \quad \mathbf{U}_{2A} \in \Re^{n \times (n - r_A)}, \quad \mathbf{V}_{1A} \in \Re^{m \times r_A}, \quad \mathbf{V}_{2A} \in \Re^{m \times (m - r_A)} \tag{3}$$

then

$$\begin{aligned} &\mathbf{U}_{1A}^{\mathrm{T}} \cdot \mathbf{U}_{1A} = \mathbf{I}_{r_A}, \quad \mathbf{U}_{1A}^{\mathrm{T}} \cdot \mathbf{U}_{2A} = \mathbf{0}, \quad \mathbf{U}_{2A}^{\mathrm{T}} \cdot \mathbf{U}_{2A} = \mathbf{I}_{n - r_A} \\ &\mathbf{V}_{1A}^{\mathrm{T}} \cdot \mathbf{V}_{1A} = \mathbf{I}_{r_A}, \quad \mathbf{V}_{1A}^{\mathrm{T}} \cdot \mathbf{V}_{2A} = \mathbf{0}, \quad \mathbf{V}_{2A}^{\mathrm{T}} \cdot \mathbf{V}_{2A} = \mathbf{I}_{m - r_A} \\ &\mathbf{U}_{2A}^{\mathrm{T}} \mathbf{A} = \mathbf{0}, \quad \mathbf{A}\mathbf{V}_{2A} = \mathbf{0}. \end{aligned} \tag{4}$$

(iii) *The algebraic equation*

$$\mathbf{A}\mathbf{x} = \mathbf{y}$$

has a solution if and only if $\mathbf{U}_{2A}^{\mathrm{T}} \mathbf{y} = \mathbf{0}$. *When this condition is satisfied, then all solutions* $\mathbf{x}$ *are of the form*

$$\mathbf{x} = \mathbf{V}_{1A} \mathbf{\Sigma}_{1A}^{-1} \mathbf{U}_{1A}^{\mathrm{T}} \mathbf{y} + \mathbf{V}_{2A} \mathbf{z}_x$$

where $\mathbf{z}_x \in \Re^{n - r_A}$ *is arbitrary.*

(iv) *Suppose* $\mathbf{A} \in \Re^{n \times m}$ *and*

$$\tilde{\mathbf{A}} = \mathbf{A} \otimes \mathbf{I}_p, \quad \mathrm{svd}(\mathbf{A}) = \{\mathbf{U}_A, \mathbf{\Sigma}_A, \mathbf{V}_A\}.$$

Then $\tilde{\mathbf{A}} \in \Re^{pn \times pm}$ *and*

$$\mathrm{svd}(\tilde{\mathbf{A}}) = \{\mathbf{U}_A \otimes \mathbf{I}_p, \mathbf{\Sigma}_A \otimes \mathbf{I}_p, \mathbf{V}_A \otimes \mathbf{I}_p\}.$$

(v) *Suppose* $\mathbf{A} \in \Re^{n \times m}$ *and*

$$\tilde{\mathbf{b}} = [\tilde{\mathbf{b}}_1^{\mathrm{T}}, \tilde{\mathbf{b}}_2^{\mathrm{T}}, \ldots, \tilde{\mathbf{b}}_n^{\mathrm{T}}]^{\mathrm{T}}, \quad \tilde{\mathbf{b}}_k = [\, b_{k1} \quad b_{k2} \quad \ldots \quad b_{kp} \,]^{\mathrm{T}} \in \Re^p. \tag{5}$$

Then the algebraic equation

$$(\mathbf{A} \otimes \mathbf{I}_p)\tilde{\mathbf{x}} = \tilde{\mathbf{b}} \tag{6}$$

has a solution if and only if the equations

$$\mathbf{A}\mathbf{x}_\ell = \mathbf{b}_\ell, \quad 1 \leqslant \ell \leqslant p \tag{7}$$

have solutions $\{\mathbf{x}_\ell \in \Re^m, 1 \leqslant \ell \leqslant p\}$ *where*

$$\mathbf{b}_\ell = [\, b_{1\ell} \quad b_{2\ell} \quad \ldots \quad b_{n\ell} \,]^{\mathrm{T}}. \tag{8}$$

If

$$\mathbf{x}_\ell = [\, x_{1\ell} \quad x_{2\ell} \quad \ldots \quad x_{m\ell} \,]^{\mathrm{T}}, \quad 1 \leqslant \ell \leqslant p \tag{9}$$

are solutions of (7), *then*

$$\tilde{\mathbf{x}} = \left[\tilde{\mathbf{x}}_1^{\mathrm{T}}, \tilde{\mathbf{x}}_2^{\mathrm{T}}, \ldots, \tilde{\mathbf{x}}_m^{\mathrm{T}}\right]^{\mathrm{T}}, \quad \tilde{\mathbf{x}}_j = [\,x_{j1} \quad x_{j2} \quad \ldots \quad x_{jp}\,]^{\mathrm{T}} \in \Re^p \tag{10}$$

is a solution of (6).

In particular, (6) *has a solution if and only if* $(\mathbf{U}_{2A}^{\mathrm{T}} \otimes \mathbf{I}_p)\tilde{\mathbf{b}} = \mathbf{0}$. *When this condition is satisfied*, *then all solutions* $\tilde{\mathbf{x}}$ *are of the form*

$$\tilde{\mathbf{x}} = (\mathbf{V}_{1A}\mathbf{\Sigma}_{1A}^{-1}\mathbf{U}_{1A}^{\mathrm{T}} \otimes \mathbf{I}_p)\tilde{\mathbf{b}} + (\mathbf{V}_{2A} \otimes \mathbf{I}_p)\tilde{\mathbf{z}}_x$$

where $\tilde{\mathbf{z}}_x \in \Re^{p(n-r_A)}$ *is arbitrary*.

3. Network representation of structures

A tensegrity structure consists of a connection of tensile components (or strings) and compressive components (or bars) in which the tension and compressive forces are directed along the strings and bars. Consequently, in this paper, the equilibrium conditions are completely specified in terms of translational forces.

We represent a tensegrity structure as an *oriented graph* (Desoer and Kuh, 1969) in real three dimensional space $\Re^3$ defined in terms of nodes and directed branches which are all represented as vectors in $\Re^3$. A *loop* is any closed path in the graph. As we shall see, the advantage of this approach is that both the magnitude and the direction of the forces are contained in vectors which can be solved using linear algebra. Thus linear algebra plays a larger role in this approach compared to the usual approach in mechanics and finite element methods using direction cosines.

In particular, suppose there are n_{b} bars and n_{s} strings for which bar contacts can only occur at the bar ends. Then in the oriented graph of a tensegrity structure, the n_{p} *nodes* consist of the ends of the bars as represented by the n_{p} vectors $\{\mathbf{p}_k\}$, and the $n_{\mathrm{b}} + n_{\mathrm{s}}$ directed branches consist of the n_{s} string branches (or vectors) $\{\mathbf{s}_n\}$ and the n_{b} bar branches (or vectors) $\{\mathbf{b}_m\}$. If there are n_{b_0} ($\leqslant n_{\mathrm{b}}$) bars which are *not* in contact with any other bar, then

$$n_{\mathrm{p}} = 2n_{\mathrm{b}_0} + \tilde{n}_{\mathrm{p}}, \quad \tilde{n}_{\mathrm{p}} \leqslant 2(n_{\mathrm{b}} - n_{\mathrm{b}_0}).$$

For example, if no two bars are in contact, then $\tilde{n}_{\mathrm{p}} = 0$ and $n_{\mathrm{p}} = 2n_{\mathrm{b}}$. Or, if $n_{\mathrm{b}} - n_{\mathrm{b}_0}$ bars all contact at a single bar end, then $\tilde{n}_{\mathrm{p}} = n_{\mathrm{b}} - n_{\mathrm{b}_0} + 1$.

Thus given a tensegrity structure consisting of n_{p} nodes, n_{b} bars and n_{s} strings, the positions of the nodes are described by the n_{p} vectors $\{\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_{n_{\mathrm{p}}}\}$, the positions of the bars are described by the n_{b} vectors $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_{n_{\mathrm{b}}}\}$, and the positions of the strings are described by the n_{s} vectors $\{\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_{n_{\mathrm{s}}}\}$.

Definition 3. The *geometry* of the tensegrity structure is defined by the tensegrity node vector $\mathbf{p} \in \Re^{3n_{\mathrm{p}}}$, the tensegrity bar vector $\mathbf{b} \in \Re^{3n_{\mathrm{b}}}$, and the tensegrity string vector $\mathbf{s} \in \Re^{3n_{\mathrm{s}}}$ where

$$\mathbf{p}^{\mathrm{T}} = [\mathbf{p}_1^{\mathrm{T}}, \mathbf{p}_2^{\mathrm{T}}, \ldots, \mathbf{p}_{n_{\mathrm{p}}}^{\mathrm{T}}], \quad \mathbf{b}^{\mathrm{T}} = [\mathbf{b}_1^{\mathrm{T}}, \mathbf{b}_2^{\mathrm{T}}, \ldots, \mathbf{b}_{n_{\mathrm{b}}}^{\mathrm{T}}], \quad \mathbf{s}^{\mathrm{T}} = [\mathbf{s}_1^{\mathrm{T}}, \mathbf{s}_2^{\mathrm{T}}, \ldots, \mathbf{s}_{n_{\mathrm{s}}}^{\mathrm{T}}]. \tag{11}$$

Definition 4. A *class k* tensegrity structure connects only *k* compressive members to a node.

***Geometric connectivity*.** Each directed branch can undergo a displacement in reaching its equilibrium state. String and bar vectors can change both their length and orientation. Node vectors can change both their length and orientation but subject to a *Law of Geometric Connectivity* which we state as follows:

The vector sum of all branch vectors in any loop is zero. (12)

These equations are in the form of a set of linear algebraic equations in the branch vectors.

Force equilibrium. In our study of tensegrity structures, we are concerned with structures in which bars sustain compressive forces and strings do not. We therefore choose to distinguish between the string (or tensile) forces $\{\mathbf{t}_n\}$ and the bar (or compressive) forces $\{\mathbf{f}_m\}$ which are defined in terms of the string and bar vectors respectively as follows.

Definition 5. Given the tensile force $\mathbf{t}_n$ in the string characterized by the string vector $\mathbf{s}_n$ and the compressive force $\mathbf{f}_n$ in the bar characterized by the bar vector $\mathbf{b}_n$, the *tensile force coefficient* $\gamma_n > 0$ and the *compressive force coefficient* $\lambda_m > 0$ are defined by

$$\mathbf{t}_n = \gamma_n \mathbf{s}_n, \quad \mathbf{f}_m = \lambda_m \mathbf{b}_m. \tag{13}$$

The forces of the tensegrity structure are defined by the external force vector $\mathbf{w} \in \Re^{3n_\mathrm{w}}$, the compression vector $\mathbf{f} \in \Re^{3n_\mathrm{b}}$, and the tension vector $\mathbf{t} \in \Re^{3n_\mathrm{s}}$ where

$$\mathbf{w}^\mathrm{T} = [\mathbf{w}_1^\mathrm{T}, \mathbf{w}_2^\mathrm{T}, \ldots, \mathbf{w}_{n_\mathrm{w}}^\mathrm{T}], \quad \mathbf{f}^\mathrm{T} = \left[\mathbf{f}_1^\mathrm{T}, \mathbf{f}_2^\mathrm{T}, \ldots, \mathbf{f}_{n_\mathrm{b}}^\mathrm{T}\right], \quad \mathbf{t}^\mathrm{T} = [\mathbf{t}_1^\mathrm{T}, \mathbf{t}_2^\mathrm{T}, \ldots, \mathbf{t}_{n_\mathrm{s}}^\mathrm{T}]. \tag{14}$$

It follows from (14) that (13) can be expressed in the form

$$\mathbf{t} = (\mathbf{\Gamma} \otimes \mathbf{I}_3)\mathbf{s}, \quad \mathbf{f} = (\mathbf{\Lambda} \otimes \mathbf{I}_3)\mathbf{b} \tag{15}$$

where

$$\mathbf{\Gamma} = \mathrm{diag}\{\gamma_1, \gamma_2, \ldots, \gamma_{n_\mathrm{s}}\}, \quad \mathbf{\Lambda} = \mathrm{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_{n_\mathrm{b}}\}. \tag{16}$$

The diagonal matrices $\{\mathbf{\Gamma}, \mathbf{\Lambda}\}$ shall be referred to as the *tensile force coefficient matrix* and *compressive coefficient force matrix*, respectively.

Force convention. Suppose each node $\mathbf{p}_k$ is subjected to compressive vector forces $\{\mathbf{f}_{mk}\}$, tensile vector forces $\{\mathbf{t}_{nk}\}$ and external force $\mathbf{w}_k$. Then the *Law for Static Equilibrium* may be stated as follows:

$$\sum_n \mathbf{t}_{nk} - \sum_m \mathbf{f}_{mk} - \mathbf{w}_k = \mathbf{0} \tag{17}$$

where a positive sign is assigned to a (tensile, compressive and external) force vector *leaving* a node, and a negative sign is assigned to a (tensile, compressive and external) force vector *entering* a node. The negative sign in (17) is a consequence of the fact that we choose to define positive force coefficients λ_n and γ_n.

From the network, it follows that components of the string vector $\mathbf{s}$ and the bar vector $\mathbf{b}$ in (11) can be written as a linear combination of components of the node vector $\mathbf{p}$. Also, if branch k is a bar which leaves node i and enters node j, then $\mathbf{b}_k = \mathbf{p}_j - \mathbf{p}_i$, whereas if branch k is a string which leaves node i and enters node j, then $\mathbf{s}_k = \mathbf{p}_j - \mathbf{p}_i$. Hence we have $\widetilde{\mathbf{C}}^\mathrm{T}\mathbf{p} = [\mathbf{s}^\mathrm{T}, \mathbf{b}^\mathrm{T}]^\mathrm{T}$ where the matrix $\widetilde{\mathbf{C}}$ consists only of block matrices of the form $\{\mathbf{0}, \pm\mathbf{I}_3\}$. In particular, if we consecutively number the $n_\mathrm{s} + n_\mathrm{b}$ branches $\{\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_{n_\mathrm{s}}, \mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_{n_\mathrm{b}}\}$ as $\{1, 2, \ldots, n_\mathrm{s}, n_\mathrm{s}+1, \ldots, n_\mathrm{s}+n_\mathrm{b}\}$, then the $3n_\mathrm{p} \times (3n_\mathrm{s} + 3n_\mathrm{b})$ matrix $\widetilde{\mathbf{C}} = [\widetilde{\mathbf{C}}_{ij}]$ is defined by

$$\widetilde{\mathbf{C}}_{ij} = \begin{cases} \mathbf{I}_3 & \text{if force vector } j \text{ enters node } i \\ -\mathbf{I}_3 & \text{if force vector } j \text{ leaves node } i \\ \mathbf{0} & \text{if force vector } j \text{ is not incident with node } i. \end{cases} \tag{18}$$

Also (i) each column of $\widetilde{\mathbf{C}}$ has exactly one block $\mathbf{I}_3$ and one block $-\mathbf{I}_3$ with all other column blocks $\mathbf{0}$, and (ii) for any row i there exists a column j such that $\widetilde{\mathbf{C}}_{ij} = \pm\mathbf{I}_3$. Specifically, the "bar connectivity" matrix $\widetilde{\mathbf{B}}$ and the "string connectivity" matrix $\widetilde{\mathbf{S}}$ form the matrix $\widetilde{\mathbf{C}}$ as follows:

$$\begin{bmatrix} \mathbf{s} \\ \mathbf{b} \end{bmatrix} = \begin{bmatrix} \widetilde{\mathbf{S}}^\mathrm{T} \\ \widetilde{\mathbf{B}}^\mathrm{T} \end{bmatrix} \mathbf{p} = \widetilde{\mathbf{C}}^\mathrm{T}\mathbf{p} \tag{19}$$

where

$$\widetilde{\mathbf{S}} \in \Re^{3n_p \times 3n_s}, \quad \widetilde{\mathbf{B}} \in \Re^{3n_p \times 3n_b}. \tag{20}$$

Similarly, we define $\widetilde{\mathbf{D}} = [\widetilde{\mathbf{D}}_{ij}]$ to be the external force incidence matrix defined by

$$\widetilde{\mathbf{D}}_{ij} = \begin{cases} \mathbf{I}_3 & \text{if external force vector } j \text{ enters node } i \\ \mathbf{0} & \text{if external force vector } j \text{ is not incident with node } i. \end{cases} \tag{21}$$

From network analysis, the law for static equilibrium and linear algebra, we have the following result.

Lemma 6. *Consider a tensegrity structure as described by the geometric conditions given by* (19). *Then the equilibrium force equations for a tensegrity structure under the external load* $\mathbf{w}$ *are*

$$\tilde{\mathbf{A}} \begin{bmatrix} \mathbf{t} \\ -\mathbf{f} \\ -\mathbf{w} \end{bmatrix} = \mathbf{0}, \quad \tilde{\mathbf{A}} \triangleq [\, \widetilde{\mathbf{S}} \quad \widetilde{\mathbf{B}} \quad \widetilde{\mathbf{D}} \,] \tag{22}$$

or equivalently

$$\widetilde{\mathbf{S}}\mathbf{t} = \widetilde{\mathbf{B}}\mathbf{f} + \widetilde{\mathbf{D}}\mathbf{w} \tag{23}$$

where

$$\widetilde{\mathbf{S}} = \mathbf{S} \otimes \mathbf{I}_3, \quad \widetilde{\mathbf{B}} = \mathbf{B} \otimes \mathbf{I}_3, \quad \widetilde{\mathbf{D}} = \mathbf{D} \otimes \mathbf{I}_3 \tag{24}$$

for some $\mathbf{S} \in \Re^{n_p \times n_s}$, $\mathbf{B} \in \Re^{n_p \times n_b}$, *and* $\mathbf{D} \in \Re^{n_p \times n_w}$.

We shall refer to $\{\mathbf{S}, \mathbf{B}, \mathbf{D}\}$ as the *string connectivity matrix*, the *bar connectivity matrix*, and load incidence matrix respectively. These incidence matrices are binary matrices whose components are $\{-1, 0, 1\}$. The computational significance of this fact is that roundoff errors in digital computers are avoided, dramatically increasing the size of problems that can be solved accurately on a digital computer.

In network analysis, the matrix $\tilde{\mathbf{A}}$ is known as the *incidence matrix*. This matrix is not the *reduced* incidence matrix since we have included the datum node which means that one block row of equations in (22) is dependent on the other rows. This fact does not cause any difficulties in subsequent developments. On the contrary, some symmetry is preserved in the algebraic equations.

Lemma 7. *Consider a tensegrity structure consisting of* $n_b(\geqslant 1)$ *bars and* $n_s(\geqslant 1)$ *strings as defined by the connectivity matrices* $\{\mathbf{S}, \mathbf{B}\}$ *in* (24). *Then*

$$\rho_B \triangleq \operatorname{rank}(\mathbf{B}) \leqslant n_b \leqslant n_p - 1.$$

In particular, if there are exactly r independent loops of only bar vectors, then $\rho_B = n_b - r$.

Proof. Each loop formed from bars connected end to end results in a linear relationship between the corresponding bar vectors. There are at most $n_p - 1$ connections between any n_p nodes in order that there be no loops. Hence, we have the required result. □

Example 8. Consider the 2-bar 4-string class 1 planar tensegrity structure illustrated in Fig. 1 with tensile force vectors $\{\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4\}$ and compressive force vectors $\{\mathbf{f}_1, \mathbf{f}_2\}$. This structure has $n_b = 2$ bars, $n_p = 4$ nodes and $n_s = 4$ strings. The Geometric Connectivity conditions (19) can be written in the form

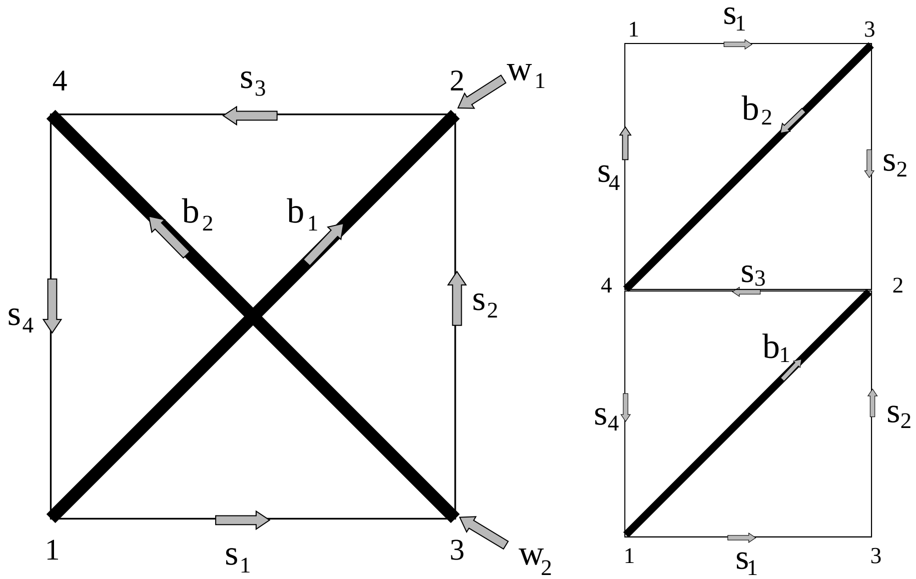

Fig. 1. 2-Bar 4-string class 1 tensegrity.

$$
\begin{aligned}
&-\mathbf{p}_1+\mathbf{p}_3=\mathbf{s}_1, \quad \mathbf{p}_2-\mathbf{p}_3=\mathbf{s}_2\\
&-\mathbf{p}_2+\mathbf{p}_4=\mathbf{s}_3, \quad \mathbf{p}_1-\mathbf{p}_4=\mathbf{s}_4\\
&-\mathbf{p}_1+\mathbf{p}_2=\mathbf{b}_1, \quad -\mathbf{p}_3+\mathbf{p}_4=\mathbf{b}_2.
\end{aligned}
\tag{25}
$$

In (19) and (24), the connectivity matrices are given by

$$
\mathbf{S}=\begin{bmatrix}-1&0&0&1\\0&1&-1&0\\1&-1&0&0\\0&0&1&-1\end{bmatrix}, \quad \mathbf{B}=\begin{bmatrix}-1&0\\1&0\\0&-1\\0&1\end{bmatrix}. \tag{26}
$$

Here $n_b=\text{rank}(\mathbf{B})=2<n_p=4$. Also, in terms of the stated force convention (17), the conditions for static equilibrium at nodes 1–4 are

$$
\left.\begin{aligned}
&-\mathbf{t}_1+\mathbf{t}_4+\mathbf{f}_1=\mathbf{0}\\
&\mathbf{t}_2-\mathbf{t}_3-\mathbf{f}_1+\mathbf{w}_1=\mathbf{0}\\
&\mathbf{t}_1-\mathbf{t}_2+\mathbf{f}_2+\mathbf{w}_2=\mathbf{0}\\
&\mathbf{t}_3-\mathbf{t}_4-\mathbf{f}_2=\mathbf{0}
\end{aligned}\right\}. \tag{27}
$$

These static equilibrium conditions and the geometric conditions can be written in the form (19)–(24), where

$$
\tilde{\mathbf{A}}=\left[\widetilde{\mathbf{S}}\middle|\widetilde{\mathbf{B}}\middle|\widetilde{\mathbf{D}}\right]\otimes\mathbf{I}_3=\left[\begin{array}{cccc|cc|cc}-1&0&0&1&-1&0&0&0\\0&1&-1&0&1&0&-1&0\\1&-1&0&0&0&-1&0&-1\\0&0&1&-1&0&1&0&0\end{array}\right]\otimes\mathbf{I}_3.
$$

Example 9. Consider the 4-bar 8-string planar class 2 tensegrity structure illustrated in Fig. 2. Now the bar connectivity **B** and the string connectivity **S** in (24) are given by

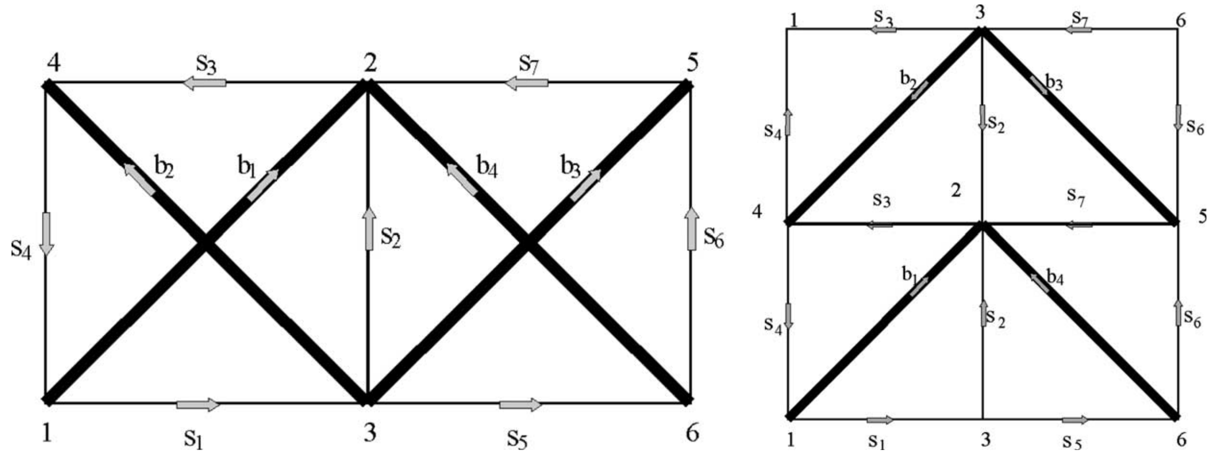

Fig. 2. 4-Bar 8-string class 2 tensegrity.

$$\mathbf{S} = \begin{bmatrix} -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

Here

$$\mathrm{rank}(\mathbf{B}) = 4 = n_\mathrm{b} < n_\mathrm{p} = 6.$$

Example 10

(a) A 3-dimensional class 1 tensegrity consisting of one stage of $n_\mathrm{b} = 3$ bars, $n_\mathrm{p} = 6$ nodes and $n_\mathrm{s} = 9$ strings is illustrated in Fig. 3. The corresponding connectivity matrices $\{\mathbf{S}, \mathbf{B}\}$ are given by

$$\mathbf{S} = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

where $\mathrm{rank}(\mathbf{B}) = 3 = n_\mathrm{b}$, $n_\mathrm{p} = 2n_\mathrm{b}$.

(b) A class 2 tensegrity structure may be derived from the structure in part (a) by replacing the strings $\{s_1, s_3, s_5\}$ by bars $\{b_4, b_5, b_6\}$ where the nodes $\{p_1, p_3, p_5\}$ now become ball joints. In this new structure, $n_\mathrm{p} = n_\mathrm{b} = n_\mathrm{s} = 6$. With $\mathbf{s} = [\mathbf{s}_2^\mathrm{T}\ \mathbf{s}_4^\mathrm{T}\ \mathbf{s}_6^\mathrm{T}\ \mathbf{s}_7^\mathrm{T}\ \mathbf{s}_8^\mathrm{T}\ \mathbf{s}_9^\mathrm{T}]^\mathrm{T}$, $\mathbf{b} = [\mathbf{b}_1^\mathrm{T}\ \mathbf{b}_2^\mathrm{T}\ \mathbf{b}_3^\mathrm{T}\ \mathbf{b}_4^\mathrm{T}\ \mathbf{b}_5^\mathrm{T}\ \mathbf{b}_6^\mathrm{T}]^\mathrm{T}$, we now have that

$$\mathbf{S} = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & -1 & 1 & 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} -1 & 0 & 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

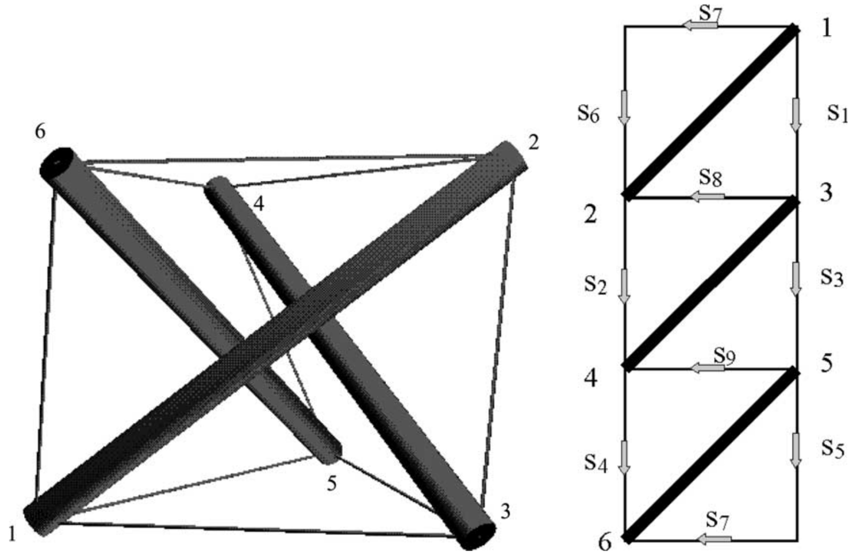

Fig. 3. 3-Bar 9-string class 1 tensegrity.

Here $\text{rank}(\mathbf{B}) = 5 = n_\text{b} - 1 < n_\text{p} = 6$. Note that since there is exactly one loop $\{\mathbf{b}_4, \mathbf{b}_5, \mathbf{b}_6\}$ of bars, it follows form Lemma 7, $\text{rank}(\mathbf{B}) = n_\text{b} - 1$.

(c) Another class 2 tensegrity structure may be derived from the structure in part (b) by replacing the string s_9 by a bar b_7. In this new structure, $n_\text{b} = 7, n_\text{p} = 6$ and $n_\text{s} = 5$. Since there are now two independent loops $\{\mathbf{b}_4, \mathbf{b}_5, \mathbf{b}_6\}$ and $\{\mathbf{b}_2, \mathbf{b}_7, \mathbf{b}_5\}$ of bars, we have from Lemma 7 that $\text{rank}(\mathbf{B}) = n_\text{b} - 2 = 5 < n_\text{p} = 6$.

We have the following result.

Theorem 11. *Suppose the $n_\text{p} \times n_\text{b}$ connectivity matrix $\mathbf{B}$ of rank ρ_B has the singular value decomposition*

$$\mathbf{B} = \begin{bmatrix} \mathbf{U}_{B1} & \mathbf{U}_{B2} \end{bmatrix} \begin{bmatrix} \mathbf{\Sigma}_{B1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{V}_{B1}^\text{T} \\ \mathbf{V}_{B2}^\text{T} \end{bmatrix}, \quad \rho_B < n_\text{b} \tag{28}$$

where $\mathbf{U}_{B1} \in \Re^{n_\text{p} \times \rho_B}$, $\mathbf{V}_{B1} \in \Re^{n_\text{b} \times \rho_B}$ or

$$\mathbf{B} = \begin{bmatrix} \mathbf{U}_{B1} & \mathbf{U}_{B2} \end{bmatrix} \begin{bmatrix} \mathbf{\Sigma}_{B1} \\ \mathbf{0} \end{bmatrix} \mathbf{V}_B^\text{T}, \quad \rho_B = n_\text{b} \tag{29}$$

where $\mathbf{U}_{B1} \in \Re^{n_\text{p} \times n_\text{b}}$, $\mathbf{V}_B \in \Re^{n_\text{b} \times n_\text{b}}$. Also, given the tensegrity node vector $\mathbf{p}$, define the coordinate transformation

$$\mathbf{p} = \widetilde{\mathbf{P}}^\text{T}\mathbf{q} \tag{30}$$

where

$$\begin{aligned} &\widetilde{\mathbf{P}} = \mathbf{P} \otimes \mathbf{I}_3, \quad \mathbf{P}^\text{T} = [\mathbf{P}_1^\text{T}, \mathbf{P}_2^\text{T}] \\ &\mathbf{P}_1 = \mathbf{\Sigma}_{B1}^{-1}\mathbf{U}_{B1}^\text{T} \in \Re^{\rho_B \times n_\text{p}}, \quad \mathbf{P}_2 = \mathbf{U}_{B2}^\text{T} \in \Re^{(n_\text{p} - \rho_B) \times n_\text{p}}. \end{aligned} \tag{31}$$

Then

(i) *In terms of the transformed tensegrity node vector $\mathbf{q}$, the tensegrity geometry is given by*

$$\mathbf{q} = \begin{bmatrix} \mathbf{q}_\text{d}^\text{T} & \mathbf{q}_\text{e}^\text{T} \end{bmatrix}^\text{T}, \quad \mathbf{s} = \widetilde{\mathbf{S}}_1^\text{T}\mathbf{q}_\text{d} + \widetilde{\mathbf{S}}_2^\text{T}\mathbf{q}_\text{e}, \quad \mathbf{q}_\text{d} = (\mathbf{V}_{B1}^\text{T} \otimes \mathbf{I}_3)\mathbf{b} \tag{32}$$

where $\mathbf{q}_d \in \Re^{3\rho_B}$, $\mathbf{q}_e \in \Re^{3(n_p-\rho_B)}$ *and*

$$\widetilde{\mathbf{S}}_1 = \mathbf{S}_1 \otimes \mathbf{I}_3, \quad \widetilde{\mathbf{S}}_2 = \mathbf{S}_2 \otimes \mathbf{I}_3$$

where

$$\mathbf{S}_1 = \mathbf{P}_1\mathbf{S} \in \Re^{\rho_B \times n_s}, \quad \mathbf{S}_2 = \mathbf{P}_2\mathbf{S} \in \Re^{(n_p-\rho_B)\times n_s}. \tag{33}$$

(ii) *The tensegrity force equilibrium is given by*

$$\begin{aligned}(\mathbf{S}_1 \otimes \mathbf{I}_3)\mathbf{t} &= (\mathbf{V}_{B1}^{\mathrm{T}} \otimes \mathbf{I}_3)\mathbf{f} + (\mathbf{D}_1 \otimes \mathbf{I}_3)\mathbf{w} \\ (\mathbf{S}_2 \otimes \mathbf{I}_3)\mathbf{t} &= (\mathbf{D}_2 \otimes \mathbf{I}_3)\mathbf{w}\end{aligned} \tag{34}$$

$$\mathbf{D}_1 = \mathbf{P}_1\mathbf{D}, \quad \mathbf{D}_2 = \mathbf{P}_2\mathbf{D}. \tag{35}$$

Proof. By Lemma 7, $\rho_B < n_p$ which means $\{\mathbf{U}_{B1}, \mathbf{U}_{B2}\}$ are well defined. Part (i) follows directly from the definition of $\widetilde{\mathbf{P}}$ with $\widetilde{\mathbf{S}}_q^{\mathrm{T}}\mathbf{q} = \mathbf{s}$, $\widetilde{\mathbf{B}}_q^{\mathrm{T}}\mathbf{q} = \mathbf{b}$ with

$$\widetilde{\mathbf{S}}_q^{\mathrm{T}} = [\widetilde{\mathbf{S}}_1^{\mathrm{T}}, \widetilde{\mathbf{S}}_2^{\mathrm{T}}], \quad \widetilde{\mathbf{B}}_q^{\mathrm{T}} = [\mathbf{V}_{B1} \otimes \mathbf{I}_3, \mathbf{0}] \tag{36}$$

with $\{\widetilde{\mathbf{S}}_1 = \widetilde{\mathbf{P}}_1\widetilde{\mathbf{S}}, \widetilde{\mathbf{S}}_2 = \widetilde{\mathbf{P}}_2\widetilde{\mathbf{S}}\}$ and $\{\widetilde{\mathbf{P}}_1\widetilde{\mathbf{B}} = \mathbf{V}_{B1}^{\mathrm{T}} \otimes \mathbf{I}_3, \widetilde{\mathbf{S}}_2 = \widetilde{\mathbf{P}}_2\widetilde{\mathbf{B}} = \mathbf{0}\}$. Part (ii) follows from the expansion of the transformed equilibrium force equation $\widetilde{\mathbf{S}}_q\mathbf{t} = \widetilde{\mathbf{B}}_q\mathbf{f} + \widetilde{\mathbf{D}}_q\mathbf{w}$, where

$$\begin{bmatrix}\widetilde{\mathbf{S}}_q & \widetilde{\mathbf{B}}_q & \widetilde{\mathbf{D}}_q\end{bmatrix} = \widetilde{\mathbf{P}}\begin{bmatrix}\widetilde{\mathbf{S}} & \widetilde{\mathbf{B}} & \widetilde{\mathbf{D}}\end{bmatrix}.$$

For notational simplicity, we assume in subsequent expressions that $\mathbf{V}_{B1} = \mathbf{V}_B$ whenever $\rho_B = n_b$. □

3.1. Class 1 structures

In a class 1 tensegrity structure, no two bars are connected, and so $n_p = 2n_b$. Without any loss of generality, we can then label the nodes of the bar $\mathbf{b}_m$ to be $\mathbf{p}_{2m}$ and $\mathbf{p}_{2m-1}$. Hence for class 1 tensegrity structures, we have

$$\mathbf{b}_m = -\mathbf{p}_{2m-1} + \mathbf{p}_{2m}, \quad m = 1, 2, \ldots, n_b. \tag{37}$$

Lemma 12. *Given the tensegrity node vector* $\mathbf{p}$ *with* (37), *bar vector* $\mathbf{b}$, *and string vector* $\mathbf{s}$, *the geometry of the class* 1 *tensegrity structure can be described by the algebraic equations*

$$\widetilde{\mathbf{B}}^{\mathrm{T}}\mathbf{p} = \mathbf{b}, \quad \widetilde{\mathbf{S}}^{\mathrm{T}}\mathbf{p} = \mathbf{s} \tag{38}$$

for some reduced $n_p \times n_s$ *connectivity matrix* $\mathbf{S}$. *The reduced* $n_p \times n_b$ *connectivity matrix* $\mathbf{B}$ *is given by*

$$\mathbf{B} = \mathbf{I}_e - \mathbf{I}_o \tag{39}$$

where odd and even node selection matrix $\mathbf{I}_o$, $\mathbf{I}_e \in \Re^{3n_p \times 3n_p}$ *are defined by*

$$\begin{aligned}\mathbf{I}_o^{\mathrm{T}} &\triangleq \mathrm{blockdiag}\{[1,0],[1,0],\ldots,[1,0],[1,0]\} \\ \mathbf{I}_e^{\mathrm{T}} &\triangleq \mathrm{blockdiag}\{[0,1],[0,1]\ldots,[0,1],[0,1]\}.\end{aligned} \tag{40}$$

Then in (29)

$$n_p = 2n_b, \quad \rho_B = n_p, \quad \mathbf{V}_B = \mathbf{I}. \tag{41}$$

The following two corollaries give two special choices for transformations.

Corollary 13. *From Theorem* 11

(i) *Let* $\mathbf{P}_1, \mathbf{P}_2 \in \Re^{n_b \times n_p}$ *in* (31) *be given by*

$$\mathbf{P}^T = \begin{bmatrix} \mathbf{P}_1^T & \mathbf{P}_2^T \end{bmatrix} = \tfrac{1}{2}[\mathbf{B} \quad \mathbf{J}] \tag{42}$$

and the inverse transformation is

$$\mathbf{P}^{-1} = 2\mathbf{P}^T = [\mathbf{B} \quad \mathbf{J}] \tag{43}$$

where $\mathbf{B}$ *is given by* (39) *and* $\mathbf{J} \in \Re^{n_b \times n_p}$ *are defined by*

$$\mathbf{J} \triangleq (\mathbf{I}_e + \mathbf{I}_o). \tag{44}$$

(ii) *The transformed coordinate* $\mathbf{q}$ *is given by*

$$\mathbf{q}^T = [\mathbf{q}_d^T, \mathbf{q}_e^T], \quad \mathbf{q}_d = \mathbf{b} \tag{45}$$

where $\mathbf{b}$ *is the bar vector and* $\mathbf{q}_e \in \Re^{3n_b}$ *is the vector of the mass center of each of the bars given by*

$$\mathbf{q}_e^T = [\mathbf{p}_{c_1}^T, \mathbf{p}_{c_2}^T, \ldots, \mathbf{p}_{c_{n_b}}^T] \tag{46}$$

where $\mathbf{p}_{c_j} \triangleq \frac{1}{2}(\mathbf{p}_{2j} + \mathbf{p}_{2j-1})$.

Corollary 14. *From Theorem* 11

(i) *Let* $\mathbf{P}_1, \mathbf{P}_2 \in \Re^{n_b \times n_p}$ *in* (31) *be given by*

$$\mathbf{P}^T = [-\mathbf{I}_o \quad \mathbf{J}] \tag{47}$$

and the inverse transformation is

$$\mathbf{P}^{-1} = [\mathbf{B} \quad \mathbf{I}_e] \tag{48}$$

where $\mathbf{B}$ *and* $\mathbf{J}$ *are given by* (39) *and* (44) *respectively, and odd and even selection matrix* $\mathbf{I}_o, \mathbf{I}_e \in \Re^{3n_p \times 3n_p}$ *are given by* (40).

(ii) *The transformed coordinate* $\mathbf{q}$ *is given by*

$$\mathbf{q}^T = [\mathbf{q}_d^T, \mathbf{q}_e^T], \quad \mathbf{q}_d = \mathbf{b} \tag{49}$$

where $\mathbf{b}$ *is the bar vector and* $\mathbf{q}_e \in \Re^{3n_b}$ *is a vector of the even nodes given by*

$$\mathbf{q}_e^T = [\mathbf{p}_2^T, \mathbf{p}_4^T, \ldots, \mathbf{p}_{n_p}^T]. \tag{50}$$

Example 15. Eqs. (25), (27) of the 2-bar 4-string tensegrity introduced in Example 8 can be written in the form (22) and (19) where $\mathbf{b}$ in (11) is already in the form (37). Then from Corollary 14, we have

$$\begin{aligned} \mathbf{S}_1 &= \begin{bmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \end{bmatrix}, \quad \mathbf{S}_2 = \begin{bmatrix} -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \\ \mathbf{P}_1 &= \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \quad \mathbf{P}_2 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{aligned} \tag{51}$$

or from Corollary 13, we have

$$\mathbf{P}_1 = \frac{1}{2}\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}, \quad \mathbf{P}_2 = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\mathbf{S}_1 = \frac{1}{2}\begin{bmatrix} 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 \end{bmatrix}, \quad \mathbf{S}_2 = \frac{1}{2}\begin{bmatrix} -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}.$$

4. Analysis of the transformed equilibrium conditions for a tensegrity structure

Definition 16. A tensegrity structure with tensile force coefficients $\{\gamma_m > 0\}$, compressive force coefficients $\{\lambda_n > 0\}$, node vector $\mathbf{p}$, string vector $\mathbf{s}$ and bar vector $\mathbf{b}$ is said to be in equilibrium if the element relationships (13), the force equations (23) and the geometric equations (19) are all satisfied.

For the remainder of this paper, we choose to use the coordinate transformation derived in Theorem 11.

Requirements for equilibrium. Given an external force vector $\mathbf{w}$, the problem of determining the geometric and force configuration of a tensegrity structure consisting of n_s strings and n_b bars in equilibrium is therefore equivalent to finding a solution $\mathbf{q}_d \in \Re^{3\rho_B}$, $\mathbf{q}_e \in \Re^{3(n_p-\rho_B)}$ of the equations

$$\mathbf{s} = (\mathbf{S}_1^T \otimes \mathbf{I}_3)\mathbf{q}_d + (\mathbf{S}_2^T \otimes \mathbf{I}_3)\mathbf{q}_e \tag{52}$$

$$\mathbf{t} = (\boldsymbol{\Gamma} \otimes \mathbf{I}_3)\mathbf{s}, \quad \boldsymbol{\Gamma} \triangleq \mathrm{diag}\{\gamma_1, \gamma_2, \ldots, \gamma_{n_s}\} \tag{53}$$

$$(\mathbf{S}_2 \otimes \mathbf{I}_3)\mathbf{t} = (\mathbf{D}_2 \otimes \mathbf{I}_3)\mathbf{w} \tag{54}$$

$$(\mathbf{V}_{B1}^T \otimes \mathbf{I}_3)\mathbf{f} = (\mathbf{S}_1 \otimes \mathbf{I}_3)\mathbf{t} - (\mathbf{D}_1 \otimes \mathbf{I}_3)\mathbf{w}, \quad \mathbf{V}_{B1}^T\mathbf{V}_{B1} = \mathbf{I} \tag{55}$$

$$\mathbf{f} = (\boldsymbol{\Lambda} \otimes \mathbf{I}_3)\mathbf{b}, \quad \boldsymbol{\Lambda} \triangleq \mathrm{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_{n_b}\} \tag{56}$$

for given matrices

$$\mathbf{S}_1 \in \Re^{\rho_B \times n_s}, \;\; \mathbf{S}_2 \in \Re^{(n_p-\rho_B)\times n_s}, \quad \mathbf{D}_1 \in \Re^{\rho_B \times n_w}, \;\; \mathbf{D}_2 \in \Re^{(n_p-\rho_B)\times n_w}. \tag{57}$$

Beyond equilibrium requirements, one might require shape constraints by requiring $\mathbf{p}$ to take on a specific set of values $\mathbf{p} = \bar{\mathbf{p}}$, where $\mathbf{p} = \widetilde{\mathbf{P}}^T\mathbf{q}$. However, in this paper, our focus is only to characterize possible equilibria, and so the freedom in choosing the nodal vector $\mathbf{p}$ will appear as free variables in the vector $\mathbf{q}_e$, as the sequel shows.

As a result of Lemma 2, conditions for the existence of solutions $\{\mathbf{q}_d \in \Re^{3\rho_B}, \mathbf{q}_e \in \Re^{3(n_p-\rho_B)}\}$ of (52)–(56) are equivalent to conditions for the existence of solutions $\{\mathbf{q}_1 \in \Re^{\rho_B}, \mathbf{q}_2 \in \Re^{n_p-\rho_B}\}$ of the equations:

$$\mathbf{s}_r = \mathbf{S}_1^T\mathbf{q}_1 + \mathbf{S}_2^T\mathbf{q}_2 \tag{58}$$

$$\mathbf{t}_r = \boldsymbol{\Gamma}\mathbf{s}_r, \quad \boldsymbol{\Gamma} \triangleq \mathrm{diag}\{\gamma_1, \gamma_2, \ldots, \gamma_{n_s}\} \tag{59}$$

$$\mathbf{S}_2\mathbf{t}_r = \mathbf{D}_2\mathbf{w}_r \tag{60}$$

$$\mathbf{V}_{B1}^T\mathbf{f}_r = \mathbf{S}_1\mathbf{t}_r - \mathbf{D}_1\mathbf{w}_r, \quad \mathbf{V}_{B1}^T\mathbf{V}_{B1} = \mathbf{I} \tag{61}$$

$$\mathbf{f}_r = \boldsymbol{\Lambda}\mathbf{b}_r, \quad \boldsymbol{\Lambda} \triangleq \mathrm{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_{n_b}\}. \tag{62}$$

4.1. Prestressed equilibrium structure

We now proceed to derive necessary and sufficient conditions for the existence of a structure in equilibrium that is prestressed in the absence of any external load; that is, $\mathbf{w} = \mathbf{0}$ in (52)–(56), or equivalently, $\mathbf{w}_r = \mathbf{0}$ in (58)–(62). Our strategy for the examination of the conditions (58)–(62) is as follows. The solution of the linear algebra problem (60) yields nonunique $\mathbf{t}_r$ which lies in the right null space of $\mathbf{S}_2$. The existence condition of $\mathbf{q}_2$ for linear algebra problem (58) yields a condition on the left null space of $\mathbf{S}_2$. Hence (58) and

(60) can be combined to obtain a unique expression for $\mathbf{t}_r$ in terms of $\mathbf{q}_1$. This is key to the main results of this paper.

We now establish necessary and sufficient conditions for a solution of equations (58)–(62) in the absence of external forces (i.e. $\mathbf{w}_r = \mathbf{0}$) by examining each of these equations in turn beginning with the solution of (58) and (60). The next result follows from Lemma 2.

Lemma 17. *Suppose*

$$\rho_2 \triangleq \operatorname{rank}(\mathbf{S}_2) \leqslant \min\{n_\mathrm{p} - \rho_B, n_\mathrm{s}\} \tag{63}$$

and let $\mathbf{S}_2$ *have the singular value decomposition* $\{\mathbf{U}, \boldsymbol{\Sigma}, \mathbf{V}\}$ *given by*

$$\mathbf{S}_2 = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^\mathrm{T} \in \Re^{(n_\mathrm{p}-\rho_B)\times n_\mathrm{s}} \tag{64}$$

where

$$\mathbf{U} = [\mathbf{U}_1, \mathbf{U}_2], \quad \boldsymbol{\Sigma} = \begin{bmatrix} \boldsymbol{\Sigma}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \mathbf{V} = [\mathbf{V}_1, \mathbf{V}_2]$$

$$\mathbf{U}_1 \in \Re^{(n_\mathrm{p}-\rho_B)\times\rho_2}, \quad \mathbf{U}_2 \in \Re^{(n_\mathrm{p}-\rho_B)\times(n_\mathrm{b}-\rho_2)}, \quad \mathbf{V}_1 \in \Re^{n_\mathrm{s}\times\rho_2}, \quad \mathbf{V}_2 \in \Re^{n_\mathrm{s}\times(n_\mathrm{s}-\rho_2)}.$$

Then a necessary and sucient condition for (58) *to have a solution* $\mathbf{q}_2 \in \Re^{n_\mathrm{p}-\rho_B}$ *is given by*

$$\mathbf{V}_2^\mathrm{T}(\mathbf{s}_r - \mathbf{S}_1^\mathrm{T}\mathbf{q}_1) = \mathbf{0}. \tag{65}$$

Furthermore, when (65) *is satisfied, all solutions* $\mathbf{q}_2$ *are of the form*

$$\mathbf{q}_2 = \mathbf{U}_1\boldsymbol{\Sigma}_{11}^{-1}\mathbf{V}_1^\mathrm{T}(\mathbf{s}_r - \mathbf{S}_1^\mathrm{T}\mathbf{q}_1) + \mathbf{U}_2\mathbf{z}_2 \tag{66}$$

where $\mathbf{z}_2 \in \Re^{n_\mathrm{b}-\rho_2}$ *is arbitrary.*

We now consider the solution of (60) when $\mathbf{w}_r = \mathbf{0}$.

Lemma 18. *When* $\mathbf{w}_r = \mathbf{0}$, *all solutions* $\mathbf{t}_r$ *of* (60) *which guarantee* (65) *are of the form*

$$\mathbf{t}_r = \mathbf{V}_2\mathbf{M}^{-1}\mathbf{V}_2^\mathrm{T}\mathbf{S}_1^\mathrm{T}\mathbf{q}_1, \quad \mathbf{M} \triangleq \mathbf{V}_2^\mathrm{T}\boldsymbol{\Gamma}^{-1}\mathbf{V}_2 \tag{67}$$

in which case

$$\mathbf{q}_2 = \mathbf{U}_1\boldsymbol{\Sigma}_{11}^{-1}\mathbf{V}_1^\mathrm{T}(\boldsymbol{\Gamma}^{-1}\mathbf{V}_2\mathbf{M}^{-1}\mathbf{V}_2^\mathrm{T} - \mathbf{I}_3)\mathbf{S}_1^\mathrm{T}\mathbf{q}_1 + \mathbf{U}_2\mathbf{z}_2 \tag{68}$$

where $\mathbf{z}_2 \in \Re^{n_\mathrm{b}-\rho_2}$ *is arbitrary.*

Proof. From (64), (65) and Lemma 2, we have $\mathbf{t}_r = \mathbf{V}_2\mathbf{z}_t$ where $\mathbf{z}_t$ is the free solution of (60). Then from (59)

$$\mathbf{V}_2^\mathrm{T}(\mathbf{s}_r - \mathbf{S}_1^\mathrm{T}\mathbf{q}_1) = \mathbf{V}_2^\mathrm{T}(\boldsymbol{\Gamma}^{-1}\mathbf{V}_2\mathbf{z}_t - \mathbf{S}_1^\mathrm{T}\mathbf{q}_1).$$

Since $\mathbf{V}_2$ has full column rank, the matrix $\mathbf{M} = \mathbf{V}_2^\mathrm{T}\boldsymbol{\Gamma}^{-1}\mathbf{V}_2$ is invertible if it exists (that is, if $\gamma_n > 0$, $n = 1, \ldots, n_\mathrm{s}$). Hence (65) is satisfied when $\mathbf{z}_t = \mathbf{M}^{-1}\mathbf{V}_2^\mathrm{T}\mathbf{S}_1^\mathrm{T}\mathbf{q}_1$, and this gives (67). □

We now consider the solution of (61) and (62) when $\mathbf{w}_r = \mathbf{0}$.

Lemma 19. *When* $\mathbf{w}_r = \mathbf{0}$, *a necessary and sucient condition for* (61) *and* (62) *to have a solution* $\mathbf{q}_1 \in \Re^{\rho_B}$ *is given by*

$$(\mathbf{X} - \mathbf{V}_{B1}^\mathrm{T}\boldsymbol{\Lambda}\mathbf{V}_{B1})\mathbf{q}_1 = \mathbf{0} \tag{69}$$

where

$$\mathbf{X} \triangleq (\mathbf{S}_1\mathbf{V}_2)\mathbf{M}^{-1}(\mathbf{S}_1\mathbf{V}_2)^{\mathrm{T}}, \quad \mathbf{M} \triangleq \mathbf{V}_2^{\mathrm{T}}\boldsymbol{\Gamma}^{-1}\mathbf{V}_2 \tag{70}$$

In particular, define

$$\rho_Y \triangleq \mathrm{rank}(\mathbf{Y}), \quad \mathbf{Y} \triangleq \mathbf{X} - \mathbf{V}_{B1}^{\mathrm{T}}\boldsymbol{\Lambda}\mathbf{V}_{B1}. \tag{71}$$

Then

(i) *when* $\rho_Y = \rho_B$, $\mathbf{q}_1 = \mathbf{0}$ *is the only solution of* (69),
(ii) *when* $\rho_Y = 0$, *any* $\mathbf{q}_1 \in \Re^{\rho_B}$ *is a solution of* (69), *and*
(iii) *when* $0 < \rho_Y < \rho_B$, *all solutions* $\mathbf{q}_1$ *are given by*

$$\mathbf{q}_1 = \mathbf{V}_{Y2}\mathbf{z}_1 \tag{72}$$

where $\mathbf{z}_1 \in \Re^{\rho_B - \rho_Y}$ *is free, and where* $\{\mathbf{U}_Y, \boldsymbol{\Sigma}_Y, \mathbf{V}_Y\}$ *is the singular value decomposition of the matrix* $\mathbf{Y} \in \Re^{\rho_B \times \rho_B}$; *that is*

$$\mathbf{U}_Y = [\mathbf{U}_{Y1}, \mathbf{U}_{Y2}], \quad \boldsymbol{\Sigma}_Y = \begin{bmatrix} \boldsymbol{\Sigma}_{Y1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \mathbf{V}_Y = [\mathbf{V}_{Y1}, \mathbf{V}_{Y2}], \quad \rho_Y = \mathrm{rank}(\mathbf{Y}) \tag{73}$$

with $\mathbf{U}_{Y1} \in \Re^{\rho_B \times \rho_Y}$, $\mathbf{V}_{Y2} \in \Re^{\rho_B \times (\rho_B - \rho_Y)}$.

Proof. From (67), (62) and (61)

$$\mathbf{S}_1\mathbf{t}_r - \mathbf{V}_{B1}^{\mathrm{T}}\mathbf{f}_r = \mathbf{S}_1\mathbf{V}_2\mathbf{M}^{-1}\mathbf{V}_2^{\mathrm{T}}\mathbf{S}_1^{\mathrm{T}}\mathbf{q}_1 - \mathbf{V}_{B1}^{\mathrm{T}}\boldsymbol{\Lambda}\mathbf{b}_r = (\mathbf{X} - \mathbf{V}_{B1}^{\mathrm{T}}\boldsymbol{\Lambda}\mathbf{V}_{B1})\mathbf{q}_1$$

where $\mathbf{X}$ is given in (70). The result then follows from the singular value decomposition of $\mathbf{X}$ after writing (69) in the form $\mathbf{X}\mathbf{q}_1 = \mathbf{V}_{B1}^{\mathrm{T}}\mathbf{f}_r$, $\mathbf{V}_{B1}^{\mathrm{T}}\mathbf{f}_r = \mathbf{V}_{B1}^{\mathrm{T}}\boldsymbol{\Lambda}\mathbf{V}_{B1}\mathbf{q}_1$. □

There are many choices of $\{\gamma_j, \lambda_k\}$ which guarantee a solution $\mathbf{q}_1 \neq \mathbf{0}$ of (69). One choice is provided in the following result.

Corollary 20. *If* $\boldsymbol{\Gamma}_1 = \gamma\mathbf{I}$, *and* $\boldsymbol{\Lambda}_1 = \lambda\mathbf{I}$, *then all solutions of* (69) *are characterized by the modal data of the* (*symmetric*) *matrix* $\mathbf{S}_1\mathbf{V}_2(\mathbf{S}_1\mathbf{V}_2)^{\mathrm{T}}$. *That is, all admissible values of* λ/γ *and* $\mathbf{q}_1$ *are eigenvalues and eigenvectors of* $\mathbf{S}_1\mathbf{V}_2(\mathbf{S}_1\mathbf{V}_2)^{\mathrm{T}}$.

Proof. From (69) and $\boldsymbol{\Gamma}_1 = \gamma\mathbf{I}$, $\boldsymbol{\Lambda}_1 = \lambda\mathbf{I}$, $\mathbf{X} = \gamma\mathbf{S}_1\mathbf{V}_2\mathbf{V}_2^{\mathrm{T}}\mathbf{S}_1^{\mathrm{T}} = \gamma\overline{\mathbf{X}}$. Then $(\gamma\bar{\mathbf{X}} - \lambda\mathbf{I})\mathbf{q}_1 = \mathbf{0}$ or $(\overline{\mathbf{X}} - \rho\mathbf{I})\mathbf{q}_1 = \mathbf{0}$, where $\rho = \lambda/\gamma$. Hence ρ and $\mathbf{q}_1$ are the eigenvalues and eigenvectors of $\overline{\mathbf{X}}$. □

We now make reference to Lemma 2, part (iv) to relate solutions of (58)–(62) as provided in Lemma 19 to solutions of (52)–(56).

Theorem 21. *Consider a tensegrity structure as defined by the geometry and force equations in the absence of external load as described by the geometric conditions*

$$(\mathbf{B}^{\mathrm{T}} \otimes \mathbf{I}_3)\mathbf{p} = \mathbf{b}, \quad (\mathbf{S}^{\mathrm{T}} \otimes \mathbf{I}_3)\mathbf{p} = \mathbf{s}, \quad \mathbf{p} \in \Re^{3n_{\mathrm{p}}}, \ \mathbf{b} \in \Re^{3n_{\mathrm{b}}}, \ \mathbf{s} \in \Re^{3n_{\mathrm{s}}}$$

and the equilibrium force equations

$$(\mathbf{S} \otimes \mathbf{I}_3)\mathbf{t} = (\mathbf{B} \otimes \mathbf{I}_3)\mathbf{f}, \quad \mathbf{t} = (\boldsymbol{\Gamma} \otimes \mathbf{I}_3)\mathbf{s}, \quad \mathbf{f} = (\boldsymbol{\Lambda} \otimes \mathbf{I}_3)\mathbf{b}$$

where for $\{\gamma_m > 0, \ \lambda_n > 0\}$,

$$\boldsymbol{\Gamma} = \mathrm{diag}\{\gamma_1, \gamma_2, \ldots, \gamma_{n_{\mathrm{s}}}\}, \quad \boldsymbol{\Lambda} = \mathrm{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_{n_{\mathrm{b}}}\}.$$

Then given any tensile force coefficients $\{\gamma_m > 0, 1 \leqslant m \leqslant n_s\}$, *there exist compressive force coefficients* $\{\lambda_n > 0, 1 \leqslant n \leqslant n_b\}$ *which define an equilibrium structure, if for some* $\mathbf{q}_1 \neq 0$, $\{\mathbf{\Lambda}, \mathbf{\Gamma}\}$ *satisfy the condition*:

$$(\mathbf{X} - \mathbf{V}_{B1}^{\mathrm{T}} \mathbf{\Lambda} \mathbf{V}_{B1})\mathbf{q}_1 = \mathbf{0}$$
$$\mathbf{X} \triangleq (\mathbf{S}_1 \mathbf{V}_2)\mathbf{M}^{-1}(\mathbf{S}_1 \mathbf{V}_2)^{\mathrm{T}}, \quad \mathbf{M} \triangleq \mathbf{V}_2^{\mathrm{T}} \mathbf{\Gamma}^{-1} \mathbf{V}_2, \tag{74}$$

where $\mathbf{V}_{B1}$ *is given by* (28) *and* (29), $\mathbf{S}_1$ *is given by* (33), *and* $\mathbf{V}_2$ *is given by* (65).

Moreover, for any $\mathbf{q}_{\mathrm{d}} \neq 0$ *which satisfies*

$$((\mathbf{X} - \mathbf{V}_{B1}^{\mathrm{T}} \mathbf{\Lambda} \mathbf{V}_{B1}) \otimes \mathbf{I}_3)\mathbf{q}_{\mathrm{d}} = \mathbf{0},$$

the nodal vector $\mathbf{p}$ *is of the form*

$$\mathbf{p} = (\mathbf{PQ} \otimes \mathbf{I}_3)[\mathbf{z}_{\mathrm{d}}^{\mathrm{T}}, \mathbf{z}_{\mathrm{e}}^{\mathrm{T}}]^{\mathrm{T}} \tag{75}$$

where $\{\mathbf{z}_{\mathrm{d}} \in \Re^{3\rho_B}, \mathbf{z}_{\mathrm{e}} \in \Re^{3(n_{\mathrm{p}} - \rho_B)}\}$ *may be arbitrarily chosen, and*

$$\mathbf{P} = [\mathbf{U}_{B1} \mathbf{\Sigma}_{B1}^{-1}, \quad \mathbf{U}_{B2}]$$
$$\mathbf{Q} \triangleq \begin{bmatrix} \mathbf{I}_3 & \mathbf{0} \\ \mathbf{L}\mathbf{V}_{Y2} & \mathbf{U}_2 \end{bmatrix}, \quad \mathbf{L} \triangleq \mathbf{U}_1 \mathbf{\Sigma}_{11}^{-1} \mathbf{V}_1^{\mathrm{T}} [\mathbf{\Gamma}^{-1} \mathbf{V}_2 \mathbf{M}^{-1} \mathbf{V}_2^{\mathrm{T}} - \mathbf{I}_3] \mathbf{S}_1^{\mathrm{T}} \tag{76}$$

where $\{\mathbf{U}_{B1}, \mathbf{\Sigma}_{B1}\}$ *are given by (28) and (29).*

The corresponding tensegrity string vector $\mathbf{s}$, *tension vector* $\mathbf{t}$, *bar vector* $\mathbf{b}$ *and compression vector* $\mathbf{f}$ *are given in terms of* $\{\mathbf{q}_{\mathrm{d}}, \mathbf{p}\}$ *by*

$$\mathbf{s} = (\mathbf{\Gamma}^{-1} \otimes \mathbf{I}_3)\mathbf{t}, \quad \mathbf{t} = (\mathbf{V}_2 \mathbf{M}^{-1} \mathbf{V}_2^{\mathrm{T}} \mathbf{S}_1^{\mathrm{T}} \otimes \mathbf{I}_3)\mathbf{q}_{\mathrm{d}}$$
$$\mathbf{b} = (\mathbf{B}^{\mathrm{T}} \otimes \mathbf{I}_3)\mathbf{p}, \quad \mathbf{f} = (\mathbf{\Lambda} \otimes \mathbf{I}_3)\mathbf{b}. \tag{77}$$

4.2. Externally loaded structures

Under the action of an external force vector $\mathbf{w}$ with component vectors $\{\mathbf{w}_j\}$ given by

$$\mathbf{w}^{\mathrm{T}} = \left[\mathbf{w}_1^{\mathrm{T}}, \mathbf{w}_2^{\mathrm{T}}, \ldots, \mathbf{w}_{n_{\mathrm{w}}}^{\mathrm{T}}\right]. \tag{78}$$

Suppose that the new equilibrium structure is assumed to be given by node vector $\mathbf{p}$, bar vector $\mathbf{b}$, string vector $\mathbf{s}$, compressive force vector $\mathbf{f}$, tensile vector $\mathbf{t}$, compressive force coefficient matrix $\mathbf{\Lambda}$ and tensile force matrix $\mathbf{\Gamma}$ as described by (52)–(56). As a result of Lemma 2, conditions for the existence of solutions $\{\mathbf{q}_{\mathrm{d}} \in \Re^{3\rho_B}, \mathbf{q}_{\mathrm{e}} \in \Re^{3(n_{\mathrm{p}} - \rho_B)}\}$ of (52)–(56) are equivalent to conditions (58)–(62) for the existence of solutions $\{\mathbf{q}_1 \in \Re^{\rho_B}, \mathbf{q}_2 \in \Re^{n_{\mathrm{p}} - \rho_B}\}$. Note that all force coefficients together with all node geometry will normally change. We now seek necessary and sufficient conditions for the externally loaded structure to be in geometric and force equilibrium. An extension of Lemmas 18 and 19 gives us the following result.

Theorem 22

(i) *All solutions* $\mathbf{t}_r$ *of* (60) *which guarantee* (65) *are of the form*

$$\mathbf{t}_r = \mathbf{V}_2 \mathbf{M}^{-1} \mathbf{V}_2^{\mathrm{T}} \mathbf{S}_1^{\mathrm{T}} \mathbf{q}_1 + \mathbf{G}_1 \mathbf{w}_r, \quad \mathbf{M} \triangleq \mathbf{V}_2^{\mathrm{T}} \mathbf{\Gamma}^{-1} \mathbf{V}_2$$
$$\mathbf{G}_1 = (\mathbf{I}_{n_{\mathrm{s}}} - \mathbf{V}_2 \mathbf{M}^{-1} \mathbf{V}_2^{\mathrm{T}} \mathbf{\Gamma}^{-1}) \mathbf{V}_1 \mathbf{\Sigma}_{11}^{-1} \mathbf{U}_1^{\mathrm{T}} \mathbf{D}_2. \tag{79}$$

(ii) *A necessary and sucient condition for* (61) *and* (62) *to have a solution* $\mathbf{b} \in \Re^{n_{\mathrm{b}}}$ *is given by*

$$(\mathbf{X} - \mathbf{V}_{B1}^{\mathrm{T}} \mathbf{\Lambda} \mathbf{V}_{B1})\mathbf{q}_1 = \mathbf{G}\mathbf{w}_r$$
$$\mathbf{U}_2^{\mathrm{T}} \mathbf{P}_2 \mathbf{w}_r = \mathbf{0} \tag{80}$$

where $\mathbf{X}$ *is given by* (70), *and*

$$\mathbf{G} \triangleq \mathbf{D}_1 - \mathbf{S}_1\mathbf{G}_1. \tag{81}$$

Proof. Since (58) is not directly dependent on $\mathbf{w}_r$, Lemma 17 applies for $\mathbf{w}_r \neq \mathbf{0}$. Now consider the solution of (60) for $\mathbf{w}_r \neq \mathbf{0}$. A necessary condition for the existence of a solution $\mathbf{t}$ is $\mathbf{U}_2^{\mathrm{T}}\mathbf{P}_2\mathbf{w}_r = \mathbf{0}$, and in this case, all solutions $\mathbf{t}_r$ are of the form

$$\mathbf{t}_r = \mathbf{V}_2\mathbf{z}_t + \mathbf{V}_1\mathbf{\Sigma}_{11}^{-1}\mathbf{U}_1^{\mathrm{T}}\mathbf{D}_2\mathbf{w}_r$$

for any $\mathbf{z}_t \in \Re^{n_b - r}$ where as in (63), r is the rank of $\mathbf{S}_2$. Now in order that condition (65) is satisfied, $\mathbf{z}_t$ must be selected such that

$$\mathbf{V}_2^{\mathrm{T}}\mathbf{\Gamma}^{-1}(\mathbf{V}_2\mathbf{z}_t + \mathbf{V}_1\mathbf{\Sigma}_{11}^{-1}\mathbf{U}_1^{\mathrm{T}}\mathbf{D}_2\mathbf{w}_r) - \mathbf{V}_2^{\mathrm{T}}\mathbf{S}_1^{\mathrm{T}}\mathbf{q}_1 = \mathbf{0}.$$

That is

$$\mathbf{z}_t = \mathbf{M}^{-1}\mathbf{V}_2^{\mathrm{T}}\mathbf{S}_1^{\mathrm{T}}\mathbf{q}_1 - \mathbf{M}^{-1}\mathbf{V}_2^{\mathrm{T}}\mathbf{\Gamma}^{-1}\mathbf{V}_1\mathbf{\Sigma}_{11}^{-1}\mathbf{U}_1^{\mathrm{T}}\mathbf{D}_2\mathbf{w}_r$$

which gives (79). From (60), (62) and (79)

$$\mathbf{S}_1\mathbf{t}_r - \mathbf{D}_1\mathbf{w}_r - \mathbf{V}_{B1}^{\mathrm{T}}\mathbf{f}_r = \mathbf{S}_1\mathbf{V}_2\mathbf{M}^{-1}\mathbf{V}_2^{\mathrm{T}}\mathbf{S}_1^{\mathrm{T}}\mathbf{q}_1 + \mathbf{S}_1\mathbf{G}_1\mathbf{w}_r - \mathbf{D}_1\mathbf{w}_r - \mathbf{V}_{B1}^{\mathrm{T}}\mathbf{\Lambda}\mathbf{b}_r$$

which gives (81). □

The first condition in (80) is a nonhomogeneous equivalent of condition (69). However it is unlikely (although not impossible) that $\mathbf{V}_{B1}^{\mathrm{T}}\mathbf{\Lambda}\mathbf{V}_{B1} = \mathbf{X}$ for $\mathbf{w}_r \neq \mathbf{0}$. Instead, is more likely that $r_Y \triangleq \rho(\mathbf{V}_{B1}^{\mathrm{T}}\mathbf{\Lambda}\mathbf{V}_{B1} - \mathbf{X})$ satisfies $0 < r_Y \leqslant n_b$, where $\{\mathbf{U}_Y, \mathbf{\Sigma}_Y, \mathbf{V}_Y\}$ is the singular value decomposition of the matrix $\mathbf{Y} \in \Re^{\rho_B \times \rho_B}$ given by (73). If $r_Y = n_b$, then $\mathbf{q}_1 = (\mathbf{\Lambda} - \mathbf{X})^{-1}\mathbf{D}\mathbf{w}_r$ is unique.

When $\mathbf{U}_2^{\mathrm{T}}\mathbf{P}_2\mathbf{w}_r = \mathbf{0}$, the solution $\mathbf{q}_1$ is of the form

$$\mathbf{q}_1 = \mathbf{V}_{Y2}\mathbf{z}_Y + \mathbf{V}_{Y1}\mathbf{\Sigma}_{b11}^{-1}\mathbf{U}_{b1}^{\mathrm{T}}\mathbf{D}\mathbf{w}_r, \quad \mathbf{U}_2^{\mathrm{T}}\mathbf{P}_2\mathbf{w}_r = \mathbf{0}$$

where $\mathbf{z}_Y \in \Re^{n_b - r_b}$ is unknown. The possibility of multiple solutions is interesting; either only one solution is possible and more information is required to determine $\mathbf{z}_Y$, or many solutions are possible. In the latter case, the particular equilibrium obtained will then depend on way in which the external load $\mathbf{w}_r$ is introduced. The structural implications of the null space condition $\mathbf{U}_2^{\mathrm{T}}\mathbf{P}_2\mathbf{w}_r = \mathbf{0}$ on the external load $\mathbf{w}_r$ would then also require a physical interpretation.

The *existence* of an equilibrium solution however requires the *second condition* in (80) on the external force $\mathbf{w}_r$ to be satisfied. In this regard, we have the following result.

Lemma 23. *For all structures* $\{\mathbf{S}, \mathbf{B}\}$, *the* $(n_b - r) \times n_s$ *matrix product* $\mathbf{U}_2^{\mathrm{T}}\mathbf{P}_2$ *is of the form*

$$\mathbf{U}_2^{\mathrm{T}}\mathbf{P}_2 = \mathbf{e}[1, 1, \ldots, 1] \tag{82}$$

for some vector $\mathbf{e}$. *Hence* $\mathbf{U}_2^{\mathrm{T}}\mathbf{P}_2\mathbf{w}_r = \mathbf{0}$ *if and only if*

$$\sum_{k=1}^{n_w} w_k = \mathbf{0}. \tag{83}$$

Proof. It follows from Lemma 2 and svd($\mathbf{S}_2$) in Lemma 17 that $\mathbf{U}_2^{\mathrm{T}}\mathbf{S}_2 = \mathbf{0}$ which from (33) implies $\mathbf{U}_2^{\mathrm{T}}\mathbf{P}_2\mathbf{S} = \mathbf{0}$. Now from Lemma 6, each column of $\mathbf{S}$ has exactly 1 and exactly −1 with all other column elements 0. Furthermore, for every ith row of $\mathbf{S}$, there exists a column j such that the ijth component of $\mathbf{S}$ is ±1. These properties of $\mathbf{S}$ then imply that $\mathbf{U}_2^{\mathrm{T}}\mathbf{P}_2$ is of the form (82) for some vector $\mathbf{e}$, and so

$$\mathbf{U}_2^{\mathrm{T}}\mathbf{P}_2\mathbf{w}_r = \mathbf{e}\sum_{k=1}^{n_{\mathrm{p}}} w_k.$$

Now if the full row rank matrix $\widetilde{\mathbf{U}}_2$ is partitioned into the block form $\widetilde{\mathbf{U}}_2^{\mathrm{T}} = [\mathbf{Z}_1, \mathbf{Z}_2, \ldots, \mathbf{Z}_{n_{\mathrm{b}}}]$ it follows from (42) that

$$\widetilde{\mathbf{U}}_2^{\mathrm{T}}\widetilde{\mathbf{P}}_2 = [\mathbf{Z}_1, \mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_2 \ldots, \mathbf{Z}_{n_{\mathrm{b}}}, \mathbf{Z}_{n_{\mathrm{b}}}]$$

which then guarantees that $\widetilde{\mathbf{U}}_2^{\mathrm{T}}\widetilde{\mathbf{P}}_2$ also has full row rank, and consequently that the matrix $\tilde{\mathbf{e}}$ is invertible. □

Condition (83) expresses the requirement that for an externally loaded tensegrity structure to be in force and geometric equilibrium, it is *necessary* (but not sufficient) that the sum of the external forces is zero.

5. Computational algorithm for equilibria

One procedure for construction of a tensegrity structure in equilibrium is provided as follows.

Step 1. Given the connectivity matrices $\mathbf{S}$ and $\mathbf{B}$ from the network topology, find a nonsingular matrix $\mathbf{P} = [\mathbf{P}_1^{\mathrm{T}}, \mathbf{P}_2^{\mathrm{T}}]$ such that $\mathbf{B}_q^{\mathrm{T}} = \mathbf{B}^{\mathrm{T}}\mathbf{P} = [\mathbf{V}_{B1}^{\mathrm{T}}, \mathbf{0}_{n_{\mathrm{b}}}]$, and calculate $\{\mathbf{S}_1 = \mathbf{P}_1\mathbf{S}, \mathbf{S}_2 = \mathbf{P}_2\mathbf{S}\}$.
Step 2. Choose $\{\gamma_m > 0\}$ and $\{\lambda_n > 0\}$ such that $\det(\mathbf{X} - \mathbf{V}_{B1}^{\mathrm{T}}\mathbf{\Lambda}\mathbf{V}_{B1}) = 0$.
Step 3a. Select suitable $\mathbf{z}_{\mathrm{d}}$ and compute $\mathbf{q}_{\mathrm{d}}$ by

$$\mathbf{q}_{\mathrm{d}} = (\mathbf{V}_{Y2} \otimes \mathbf{I}_3)\mathbf{z}_{\mathrm{d}}.$$

Step 3b. When the bar connectivity matrix $\mathbf{B}$ has full rank $\rho_B = n_{\mathrm{b}}$ (i.e. no loops of bar vectors), then one can select suitable $\mathbf{z}_{\mathrm{d}}$ and compute $\mathbf{b}$ by

$$\mathbf{b} = (\mathbf{V}_{B1}\mathbf{V}_{Y2} \otimes \mathbf{I}_3)\mathbf{z}_{\mathrm{d}}.$$

Step 4. Select $\mathbf{z}_{\mathrm{e}}$ and compute the node vector $\mathbf{p}$ from (75).
Step 5. Compute $\{\mathbf{t}, \mathbf{s}, \mathbf{f}, \mathbf{b}\}$ from

$$\begin{aligned} &\mathbf{s} = (\mathbf{\Gamma}^{-1} \otimes \mathbf{I}_3)\mathbf{t}, \quad \mathbf{b} = (\mathbf{B}^{\mathrm{T}} \otimes \mathbf{I}_3)\mathbf{p}, \quad \mathbf{f} = (\mathbf{\Lambda} \otimes \mathbf{I}_3)\mathbf{b} \\ &\mathbf{t} = (\mathbf{V}_2\mathbf{M}^{-1}\mathbf{V}_2^{\mathrm{T}}\mathbf{S}_1^{\mathrm{T}} \otimes \mathbf{I}_3)\mathbf{q}_{\mathrm{d}} + (\mathbf{D}_1 \otimes \mathbf{I}_3)\mathbf{w} \\ &\mathbf{D}_1 = (\mathbf{I}_{n_{\mathrm{s}}} - \mathbf{V}_2\mathbf{M}^{-1}\mathbf{V}_2^{\mathrm{T}}\mathbf{\Gamma}^{-1})\mathbf{V}_1\mathbf{\Sigma}_{11}^{-1}\mathbf{U}_1^{\mathrm{T}}\mathbf{P}_2. \end{aligned} \tag{84}$$

6. Illustrated examples

We now illustrate the construction procedure for a simple tensegrity structure.

Example 24. A general force configuration for the class 1 tensegrity structure in Example 8 with $\mathbf{w}_1 = \mathbf{w}_2 = \mathbf{0}$ will be investigated in this section. Suppose the force coefficient matrices are given by $\mathbf{\Lambda} = \mathrm{diag}\{\lambda_1, \lambda_2\}$ and $\mathbf{\Gamma} = \mathrm{diag}\{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$.

Step 1. The connectivity matrices $\mathbf{S}$, $\mathbf{B}$ and the coordinate transformation $\mathbf{P} = [\mathbf{P}_1^{\mathrm{T}}, \mathbf{P}_2^{\mathrm{T}}]$ are given in Example 8. Since $\mathbf{B}$ is of full column rank matrix and $\mathbf{V}_2$ spans the null space of $\mathbf{S}_2$, we compute $\mathbf{V}_{B1} = \mathbf{I}_2$ and

$$\mathbf{V}_2 = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Step 2. Choose $\{\gamma_m > 0\}$ and $\{\lambda_n > 0\}$ such that $\det(\mathbf{X} - \mathbf{V}_{B1}^{\mathrm{T}}\mathbf{\Lambda}\mathbf{V}_{B1}) = 0$, where

$$\mathbf{X} = \mathbf{S}_1\mathbf{V}_2(\mathbf{V}_2^{\mathrm{T}}\mathbf{\Gamma}^{-1}\mathbf{V}_2)^{-1}\mathbf{V}_2^{\mathrm{T}}\mathbf{S}_1^{\mathrm{T}} = \frac{1}{\gamma_1+\gamma_2+\gamma_3+\gamma_4}\begin{bmatrix} (\gamma_2+\gamma_3)(\gamma_4+\gamma_1) & (-\gamma_3\gamma_1+\gamma_2\gamma_4) \\ (-\gamma_3\gamma_1+\gamma_2\gamma_4) & (\gamma_3+\gamma_4)(\gamma_2+\gamma_1) \end{bmatrix}.$$

In all choices for $\{\lambda_n\}$ that led to the rank of $(\mathbf{X} - \mathbf{\Lambda})$ having rank 1, the 4×2 matrix $\mathbf{V}_{Y2}$ is of the form $\mathbf{V}_{X2}^{\mathrm{T}} = [\mathbf{A}_1^{\mathrm{T}}, \pm\mathbf{A}_1^{\mathrm{T}}]$ in (72); that is the two bar vectors $\{\mathbf{b}_1, \mathbf{b}_2\}$ are always parallel, so the equilibrium structure is one dimensional with $\{\mathbf{p}_1 = \mathbf{p}_3, \mathbf{p}_2 = \mathbf{p}_4\}$. Hence for a two-dimensional structure, $\mathbf{X} - \mathbf{\Lambda}$ must have rank zero. This requires

$$\begin{aligned} \gamma_4 &= \frac{\gamma_3\gamma_1}{\gamma_2} \\ \lambda_1 &= \frac{(\gamma_2+\gamma_3)(\gamma_4+\gamma_1)}{\gamma_2+\gamma_3+\gamma_4+\gamma_1} \\ \lambda_2 &= \frac{(\gamma_3+\gamma_4)(\gamma_2+\gamma_1)}{\gamma_2+\gamma_3+\gamma_4+\gamma_1} \end{aligned} \tag{85}$$

where γ_1, γ_2, and γ_3 are free positive constants. We choose $\{\gamma_k = 1; k = 1,2,3\}$ and then $\mathbf{\Gamma} = \mathbf{I}_4$. It follows that $\mathbf{X} = \mathbf{I}_2$ and $\mathbf{\Lambda} = \mathbf{I}_2$ satisfy condition (74) in Theorem 21.

Step 3b. When the bar connectivity matrix $\mathbf{B}$ has full rank $\rho_B = n_{\mathrm{b}}$ (i.e. no loops of bar vectors), then one can select suitable $\mathbf{z}_{\mathrm{d}}$ and compute $\mathbf{b}$ by $\mathbf{b} = (\mathbf{V}_{B1}\mathbf{V}_{Y2} \otimes \mathbf{I}_3)\mathbf{z}_{\mathrm{d}}$. Since we choose γ and λ such that $\mathbf{X} - \mathbf{\Lambda} = \mathbf{0}$, $\mathbf{V}_{Y2} = \mathbf{I}$; that is the bar vector is arbitrary. Let us choose $\mathbf{b}_1 = [2,0]^{\mathrm{T}}$, $\mathbf{b}_2 = [0,2]^{\mathrm{T}}$.

Step 4. Select $\mathbf{z}_{\mathrm{e}}$ and compute the node vector $\mathbf{p}$ from (75).

The nodes $\{\mathbf{p}_1 = [-1,0]^{\mathrm{T}}, \mathbf{p}_2 = [1,0]^{\mathrm{T}}, \mathbf{p}_3 = [0,-1]^{\mathrm{T}}, \mathbf{p}_4 = [0,1]^{\mathrm{T}}\}$ define an equilibrium solution from (75) setting $\mathbf{z}_{\mathrm{e}}$ is zero. When we set $\mathbf{z}_{\mathrm{e}} = [1,1]^{\mathrm{T}}$, we obtain the nodal vector

$$\mathbf{p}_1 = \begin{bmatrix} -0.6464 \\ 0.3536 \end{bmatrix}, \quad \mathbf{p}_2 = \begin{bmatrix} 1.3536 \\ 0.3536 \end{bmatrix}, \quad \mathbf{p}_3 = \begin{bmatrix} 0.3536 \\ -0.6464 \end{bmatrix}, \quad \mathbf{p}_4 = \begin{bmatrix} 0.3536 \\ 1.3536 \end{bmatrix}.$$

This choices of $\mathbf{z}_{\mathrm{e}}$ only translate the geometric center of the structure from $[0,0]^{\mathrm{T}}$ to $0.3536[1,1]^{\mathrm{T}}$, since all force coefficients and $\mathbf{b}$ have been specified.

Step 5. Compute $\{\mathbf{t}, \mathbf{s}, \mathbf{f}\}$ from (84).

Example 25. Consider the $(3,9;3)$ class 1 tensegrity structure defined in Example 10. A symmetrical force configuration will be investigated with equal bar force coefficients $\{\lambda_1 = \lambda_2 = \lambda_3 = \lambda\}$, equal 'base' string force coefficients $\{\gamma_1 = \gamma_2 = \gamma_3 \triangleq \gamma_{\mathrm{b}}\}$, equal 'top' string coefficients $\{\gamma_4 = \gamma_5 = \gamma_6 \triangleq \gamma_{\mathrm{t}}\}$, and equal vertical string coefficients $\{\gamma_7 = \gamma_8 = \gamma_9 \triangleq \gamma_{\mathrm{v}}\}$. Then

$$\mathbf{X} = \mathbf{S}_1\mathbf{V}_2(\mathbf{V}_2^{\mathrm{T}}\mathbf{\Gamma}^{-1}\mathbf{V}_2)^{-1}\mathbf{V}_2^{\mathrm{T}}\mathbf{S}_1^{\mathrm{T}} = \frac{1}{(2\gamma_{\mathrm{b}}^2 + 8\gamma_{\mathrm{b}}\gamma_{\mathrm{t}} + 6\gamma_{\mathrm{b}}\gamma_{\mathrm{v}} + 6\gamma_{\mathrm{v}}\gamma_{\mathrm{t}} + 2\gamma_{\mathrm{t}}^2 + 3\gamma_{\mathrm{v}}^2)\gamma_{\mathrm{t}}^3\gamma_{\mathrm{b}}^3\gamma_{\mathrm{v}}^3}[\mathbf{X}_1 \quad \mathbf{X}_2 \quad \mathbf{X}_3],$$

where

$$\mathbf{X}_1 = \begin{bmatrix} (\gamma_{\mathrm{t}}+\gamma_{\mathrm{v}}+\gamma_{\mathrm{b}})(\gamma_{\mathrm{v}}^2 + 3\gamma_{\mathrm{v}}\gamma_{\mathrm{t}} + 3\gamma_{\mathrm{b}}\gamma_{\mathrm{v}} + 4\gamma_{\mathrm{b}}\gamma_{\mathrm{t}} + \gamma_{\mathrm{b}}^2 + \gamma_{\mathrm{t}}^2) \\ -\gamma_{\mathrm{b}}\gamma_{\mathrm{v}}\gamma_{\mathrm{t}} - \gamma_{\mathrm{b}}\gamma_{\mathrm{t}}^2 - 4\gamma_{\mathrm{b}}^2\gamma_{\mathrm{t}} - 3\gamma_{\mathrm{b}}^2\gamma_{\mathrm{v}} - \gamma_{\mathrm{b}}^3 + \gamma_{\mathrm{v}}\gamma_{\mathrm{t}}^2 + 2\gamma_{\mathrm{t}}\gamma_{\mathrm{v}}^2 + \gamma_{\mathrm{v}}^3 \\ -4\gamma_{\mathrm{b}}\gamma_{\mathrm{t}}^2 - \gamma_{\mathrm{b}}\gamma_{\mathrm{v}}\gamma_{\mathrm{t}} - \gamma_{\mathrm{b}}^2\gamma_{\mathrm{t}} + 2\gamma_{\mathrm{b}}\gamma_{\mathrm{v}}^2 + \gamma_{\mathrm{b}}^2\gamma_{\mathrm{v}} - 3\gamma_{\mathrm{v}}\gamma_{\mathrm{t}}^2 - \gamma_{\mathrm{t}}^3 + \gamma_{\mathrm{v}}^3 \end{bmatrix}$$

$$\mathbf{X}_2 = \begin{bmatrix} -\gamma_b\gamma_v\gamma_t - \gamma_b\gamma_t^2 - 4\gamma_b^2\gamma_t - 3\gamma_b^2\gamma_v - \gamma_b^3 + \gamma_v\gamma_t^2 + 2\gamma_t\gamma_v^2 + \gamma_v^3 \\ 10\gamma_b\gamma_v\gamma_t + 4\gamma_b\gamma_t^2 + 7\gamma_b^2\gamma_t + 5\gamma_b\gamma_v^2 + 6\gamma_b^2\gamma_v + \gamma_b^3 + 2\gamma_v\gamma_t^2 + 3\gamma_t\gamma_v^2 + \gamma_v^3 \\ -3\gamma_b\gamma_t^2 - \gamma_b\gamma_v\gamma_t - \gamma_v\gamma_t^2 + \gamma_t\gamma_v^2 - 3\gamma_b^2\gamma_t + \gamma_b\gamma_v^2 - \gamma_b^2\gamma_v + \gamma_v^3 \end{bmatrix}$$

$$\mathbf{X}_3 = \begin{bmatrix} -4\gamma_b\gamma_t^2 - \gamma_b\gamma_v\gamma_t - \gamma_b^2\gamma_t + 2\gamma_b\gamma_v^2 + \gamma_b^2\gamma_v - 3\gamma_v\gamma_t^2 - \gamma_t^3 + \gamma_v^3 \\ -3\gamma_b\gamma_t^2 - \gamma_b\gamma_v\gamma_t - \gamma_v\gamma_t^2 + \gamma_t\gamma_v^2 - 3\gamma_b^2\gamma_t + \gamma_b\gamma_v^2 - \gamma_b^2\gamma_v + \gamma_v^3 \\ 6\gamma_v\gamma_t^2 + 5\gamma_t\gamma_v^2 + 10\gamma_b\gamma_v\gamma_t + 7\gamma_b\gamma_t^2 + 4\gamma_b^2\gamma_t + \gamma_t^3 + \gamma_v^3 + 3\gamma_b\gamma_v^2 + 2\gamma_b^2\gamma_v \end{bmatrix}.$$

Now we need to choose the force coefficients such that

$$\det(\mathbf{X} - \mathbf{V}_{B1}^{\mathrm{T}}\mathbf{\Lambda}\mathbf{V}_{B1}) = 0.$$

Since bar connectivity matrix is of full column rank, $\mathbf{V}_{B1} = \mathbf{I}$ and

$$\det(\mathbf{X} - \mathbf{\Lambda}) = -\frac{(\lambda - \gamma_v)}{2\gamma_b^2 + 8\gamma_b\gamma_t + 6\gamma_b\gamma_v + 6\gamma_v\gamma_t + 2\gamma_t^2 + 3\gamma_v^2}\lambda_{\text{second}} = 0$$

where

$$\begin{aligned}\lambda_{\text{second}} \triangleq (&-2\gamma_t^3\lambda + 2\gamma_v^2\gamma_t^2 + 2\gamma_v\gamma_t^3 + 6\gamma_b^3\gamma_t + 15\gamma_b^2\gamma_t^2 + 2\gamma_b^2\gamma_v^2 + 6\gamma_b\gamma_t^3 + 2\gamma_b^3\gamma_v - 6\gamma_v^2\gamma_t\lambda - 10\gamma_v\gamma_t^2\lambda - 16\gamma_b^2\gamma_t\lambda \\ &+ 2\gamma_b^2\lambda^2 - 2\gamma_b^3\lambda - 10\gamma_b^2\gamma_v\lambda + 6\gamma_b\lambda^2\gamma_v - 6\gamma_b\lambda\gamma_v^2 + 8\gamma_b\lambda^2\gamma_t - 16\gamma_b\lambda\gamma_t^2 - 22\gamma_b\lambda\gamma_v\gamma_t \\ &+ 2\lambda^2\gamma_t^2 + 3\lambda^2\gamma_v^2 + 6\lambda^2\gamma_v\gamma_t + 16\gamma_b\gamma_t^2\gamma_v + 8\gamma_b\gamma_t\gamma_v^2 + 16\gamma_b^2\gamma_v\gamma_t).\end{aligned}$$

Since smaller rank of $(\mathbf{X} - \mathbf{\Lambda})$ yields more freedom for the choice of $\mathbf{b}$, we choose $\gamma_v = \lambda$. Next evaluating the second term when $\gamma_v = \lambda$, we have

$$\begin{aligned}\lambda_{\text{second}}|_{\gamma_v=\lambda} &= -6\lambda^2\gamma_t^2 - 6\gamma_b^2\lambda^2 + 6\gamma_b\gamma_t^3 + 6\gamma_b^3\gamma_t + 15\gamma_b^2\gamma_t^2 - 6\gamma_b\lambda^2\gamma_t + 3\lambda^4 \\ &= (\lambda^2 - \gamma_t\gamma_b - 2\gamma_b^2)(\lambda^2 - \gamma_t\gamma_b - 2\gamma_t^2).\end{aligned}$$

We conclude $\lambda = \sqrt{\gamma_t\gamma_b + 2\gamma_b^2}$ or $\lambda = \sqrt{\gamma_t\gamma_b + 2\gamma_t^2}$, since λ should be positive.

When we apply $\lambda = \sqrt{\gamma_t\gamma_b + 2\gamma_t^2}$,

$$\mathbf{X} - \mathbf{\Lambda} = \frac{(\gamma_b - \gamma_t)(\gamma_b + \gamma_t)}{2\gamma_b^2 + 6\gamma_b\gamma_{bt} + 11\gamma_b\gamma_t + 8\gamma_t^2 + 6\gamma_t\gamma_{bt}}\overline{\mathbf{X}}.$$

where $\gamma_{bt} \triangleq \sqrt{\gamma_t(\gamma_b + 2\gamma_t)}$ and

$$\overline{\mathbf{X}} = \begin{bmatrix} \gamma_b + 2\gamma_{bt} + 3\gamma_t & -\gamma_b - 4\gamma_t - 3\gamma_{bt} & \gamma_t + \gamma_{bt} \\ -\gamma_b - 4\gamma_t - 3\gamma_{bt} & \gamma_b + 4\gamma_{bt} + 6\gamma_t & -2\gamma_t - \gamma_{bt} \\ \gamma_t + \gamma_{bt} & -2\gamma_t - \gamma_{bt} & \gamma_t \end{bmatrix}.$$

Note that the rank of the matrix $\overline{\mathbf{X}}$ is 1. An interesting case is when $\gamma_t = \gamma_b$. In this case, an equilibrium solution with $\mathbf{\Lambda} = \mathbf{X}$ is provided by $\lambda = \gamma_v = \sqrt{3}\gamma_t$ for all choices of $\{\gamma_t\}$. Hence the bar vector can be freely chosen in this case. When $\gamma_t = \gamma_b = 1$ and $\lambda = \gamma_v = \sqrt{3}$, the structural shape is the prism given in Fig. 3.

7. Conclusion

This paper characterizes the static equilibria of tensegrity structures. Analytical expressions are derived for the equilibrium condition of a tensegrity structure in terms of member force coefficients and string and bar connectivity information. We use vectors to describe each element (bars and tendons), eliminating the need to use direction cosines and the subsequent transcendental functions that follow their use. By enlarging the vector space in which we characterize the problem, the mathematical structure of the equations admit treatment by linear algebra methods, for the most part. This reduces the study of a significant portion of the tensegrity equilibria to a series of linear algebra problems. Our results characterize the equilibria conditions of tensegrity structures in terms of a very small number of variables since the necessary and sufficient conditions of the linear algebra treatment has eliminated several of the original variables. This formulation offers insight and identifies the free parameters that may be used to achieve desired structural shapes. Since all conditions are necessary and sufficient, these results can be used in the design of any tensegrity structure. Special insightful properties are available in the special case when one designs a tensegrity structure so that all strings have the same force per unit length (γ), and all bars have the same force per unit length (λ). In this case, all admissible values of λ/γ are the discrete set of eigenvalues of a matrix given in terms of only the string connectivity matrix. Furthermore the only bar vectors which can be assigned are eigenvectors of the same matrix. Future papers will integrate these algorithms into software to make these designs more efficient.

Acknowledgements

Part of this research was conducted by Dynamic Systems Research Inc. under the sponsorship of the Jet Propulsion Laboratory, technical monitors Dr. Marco Quadrelli and Fred Hadaegh.

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